STUDIES ON SOME NONLINEAR ASPECTS OF PLASMAS

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CONTENTS

FOREWORD

This thesis comprises four parts.

In Part 1, a statistical theory of the finite-amplitude effects in electron plasma oscillations is developed. This is the field of study whose subject is more commonly known by the terms turbulent phenomena. The phenomena of turbulent fluctua in plasmas are still very poorly understood today, although they have a grave consequence on the confinement of thermonuclear pla Therefore, it is urgently hoped that the results obtained would a breakthrough into the domain of these extremely intractable pr

Part 1 consists of two chapters.

In the first chapter, the method of redundant variables f many-body problems is fully exploited in order to analyse, by me of collective coordinates, the macroscopic turbulent electric fi fluctuations in electron plasmas. The theory is based on a non perturbational technique, which utilizes the statistical stabili of the Gaussian distribution of random fluctuations. In this spirit, an attempt is made to clarify the stationary phase distr bution of large-amplitude long-wavelength longitudinal electric field fluctuations, neglecting the effects due to thermal motion of individual electrons; but the most essential feature of nonli interactions between distinct modes is entirely retained.

In the second chapter of Part 1, an investigation is made the energy distribution, in a wave number space, of the macrosco turbulent electric field fluctuations in electron plasmas. Oscillatory longitudinal electric fields are expressed by means of collective coordinates as a system of coupled oscillators with non-linear mode couplings. The model under consideration is such an electron plasma in which finite amplitude plasma oscillations are strongly excited through a long wavelength region and are thus sustaining a quasi-stationary state. An attempt is made to determine the energy spectral distribution in the wave number space approximately, by checking on frequency-shift relations, but without looking into the mechanism of thermalizing processes. It is found that the curve of the energy distribution rises linearly with $\underline{k}$ up to a critical wave number. Beyond the critical wave number which depends on the intensity of the turbulent fluctuations, the curve rapidly falls off to zero as an inverse square of $\underline{k}$.

Part 2 is mostly concerned with the diffusion process of plasmas across an externally-applied magnetic field. Estimation is made of the diffusion of guiding centers of charged particles across the magnetic field.

In Part 3 a tricky method is introduced in order to find proper starting values for the numerical solution of simultaneous nonlinear algebraic equations. The method is a new version of the technique which is more commonly called the Monte Carlo method. Particle physics analogue is fully utilized in introducing certain hypothetical potential fields which are constructed from the given equations. Confirmation of the theoretical prediction was carried out on a digital computer.

In Part 4 a method of the instability analysis of weakly-

ionized plasmas was applied to describe the formation and orientation of the equatorial (ionospheric) sporadic E layers.
The theory was found satisfactory in interpreting various observed facts.

In the end, a summary is given of the results obtained in the present thesis.

The theoretical results stated in Chapters 1 and 2 of Part 1 have already been published in the Progress of Theoretical Physics, Volume 28, Number 3, September 1962 and in the same journal, Volume 30, Number 1, July 1963, respectively. Part of the contents of Part 2 has also been published in the Japanese Journal of Thermonuclear Research (Kakuyugo Kenkyu), Volume 3, Number 6, December 1959. As for Part 3, none of the results are not published yet, but are scheduled to be contributed to some authorized journal elsewhere. Contents of Part 4 are now in press and will appear in the Report of Ionosphere and Space Research in Japan in the near future as a joint work with Profs. K. Maeda and H. Maeda.

PART 1.

A STATISTICAL THEORY OF THE FINITE-AMPLITUDE EFFECTS IN ELECTRON PLASMA OSCILLATIONS

Chapter 1. A theory of the turbulent electric field fluctuations I. --- Mean rate of fluctuations.

§1.1 Introduction

The plasma which will be considered hereafter is the electron plasma, in which ions are assumed to form a uniform background of positive charge to keep the plasma electrically neutral as a whole. Even such an idealized plasma is capable of a wide variety of oscillatory motions. We shall assume that neither external electric field nor magnetic field is applied during the evolution of the phenomena, and ignore the existence of radiation fields. These restrictions may be removed with no great conceptual difficulties.

Suppose a sudden impulsive disturbance, say, an impulsive and strong electric field, is initially introduced into a plasma. The plasma will then be subject to turbulent electric field oscillations. More exactly speaking, when such a large-amplitude electron plasma oscillation with a certain wavelength (which may be of the order of the characteristic dimension of the plasma) occurs, the oscillating electric field energy is split into other degrees of freedom through nonlinear interactions. Thus, at the cost of the decrease of the energy in the original wave mode, various other finite-amplitude modes are excited and finally, after such a cascade process, the so-called <u>turbulent</u> longitudinal electric field fluctuations are developed.

Using a method similar to that of the theory of the hydrodynamic turbulence, we analyze such a turbulent electric field fluctuation into various constituent longitudinal waves, each of which is due to an organized motion of the plasma. Frequent nonlinear interactions cause various elementary waves to grow or decay, thus the resultant state made up from such an aggregate of multi-component system of waves appears to be almost random in nature.

It is believed that the best approach to this problem is through the theory of collective behavior in the many-body problem, since the long-wavelength electric field fluctuations of the present concern are not due to individual (thermal) motion of the plasma particles. We employ the so-called method of redundant variables. Namely, we divide the total Hamiltonian for the electron plasma with Coulomb interaction into three parts: H_c (collective part), H_{int} (particle-versus-collective interaction) and H_p (particle part). Particle motions can be described as if they occur in an external field due to H_c ; but, to make the quantities self-consistent, another condition of restraint is imposed on the relation between the collective motion and the particle behavior. By this procedure, we can derive rigorous equations concerning both the phase distribution of the fluctuation and the particle aspect of the plasma.

We next subdivide the wave number region into

$$k_o < k_c < k_d < \infty,$$

where k_d is the wave number corresponding to the Debye wave-length; k_c is a critical wave number which can be defined by k_o

and the ratio of the total fluctuation energy to the thermal particle energy of the plasma; and k_0 is the wave number corresponding to the characteristic dimension of the plasma system. In the region (k_c, k_d) the collective behavior and the particle aspect counterbalance each other; but in the long wavelength region (k_0, k_c), the particle aspect gives no appreciable effect and the mode-to-mode coupling nonlinear effects are of vital concern. This is the region which corresponds to the inertial subrange of the statistical hydrodynamic turbulence where the mode coupling determines its statistical properties and even the energy dissipation plays no important role. In this region we cannot neglect the interchange of energy between the various modes.

Next we examine the case in which the turbulent electric field fluctuations are spatially homogeneous and stationary in time. It can be shown by a calculation of the characteristic function that the phase distribution is a normal distribution in an asymptotic sense and that the energy equipartition holds approximately in the region (k_0, k_c). Thus by means of the quasi-normality of the phase distribution we are able to obtain a closed system of differential equations, from which the mean speed of the phase variation of the electric field fluctuations is derived. The lifetime of the irregularity in the electron density is also calculated. The units are e. s. u.

§1.2 Method of redundant variables[1)]

For an electron plasma, where all positive ions are assumed to be smeared-out background, the total Hamiltonian of the system composed of N electrons in a volume V is given as follows:

$$H = \sum_j (\vec{p}_j^{\,2}/2m) + \frac{1}{2}\sum_{j \neq n} U(r_{jn}), \qquad (1.1)$$

where m is the electron mass, $\vec{p}_j$ the momentum of the j^{th} electron, r_{jn} the distance between the j^{th} and n^{th} electrons and the last term represents the Coulomb potential energy of the system.

Now let us introduce a set of canonical variables $Q_{\vec{k}}$, $P_{\vec{k}}$, where $\vec{k}$ belongs to a certain set of ν wave numbers $\{\vec{k}\}$. The set $\{\vec{k}\}$ comprises all the wave numbers, both $+\vec{k}$ and $-\vec{k}$, its magnitude k $(=|\vec{k}|)$ being

$$k_0 < k < k_d, \qquad (1.2)$$

where k_0 corresponds to the linear dimension under consideration, and k_d to the Debye wavelength.

As the variables $Q_{\vec{k}}$ and $P_{\vec{k}}$ are not included in the original Hamiltonian, the possible orbit of the system in $(3N+\nu)$-dimensional space must comply with, say,

$$Q_{\vec{k}} = constant = 0. \qquad (1.3)$$

Next, let us carry out a canonical transformation with a generating function defined by

$$\Omega = \sum_{\{\vec{k}\}} P'_{\vec{k}} (Q_{\vec{k}} + \xi_{\vec{k}}) + \sum_j (\vec{p}_j{}' \cdot \vec{x}_j), \qquad (1.4)$$

where $\xi_{\vec{k}}$ is defined to be

$$\xi_{\vec{k}} = (Nk)^{-1} \sum_j \exp(-i\vec{k}\cdot\vec{x}_j) \quad (= \xi^*_{-\vec{k}}). \qquad (1.5)$$

In the above, the $\vec{x}_j$ is the position vector of the j^{th} electron, and the symbol "*" represents the complex conjugate of the quantity to which it is attached.

Now, general relations in canonical transformation theory

$$\left.\begin{aligned} Q'_{\vec{k}} &= \partial\Omega/\partial P'_{\vec{k}}, \quad \vec{x}_j{}' = \partial\Omega/\partial\vec{p}_j{}'; \\ P_{\vec{k}} &= \partial\Omega/\partial Q_{\vec{k}}, \quad \vec{p}_j = \partial\Omega/\partial\vec{x}_j \end{aligned}\right\} \qquad (1.6)$$

yield the following relation:

$$\left.\begin{aligned} Q'_{\vec{k}} &= \xi_{\vec{k}}, \\ P_{\vec{k}} &= P'_{\vec{k}}, \\ \vec{x}_j{}' &= \vec{x}_j, \\ \vec{p}_j &= \vec{p}_j{}' - \sum_{\{\vec{k}\}} \frac{\vec{k}}{Nk} P'_{\vec{k}} \exp(-i\vec{k}\cdot\vec{x}_j). \end{aligned}\right\} \qquad (1.7)$$

The potential part contained in the original Hamiltonian can be expanded rigorously in terms of our collective coordinate (i.e. without having recourse to the random phase approximation):

$$\frac{1}{2}\sum_{j\neq n} U(t_{jn}) = U_0(\vec{x}_j) + \frac{1}{2}N^2\sum_{\{\vec{k}\}} k^2 U(k)\,\xi_{\vec{k}}\,\xi_{-\vec{k}}\,, \qquad (1.8)$$

where

$$U_0(\vec{x}_j) = \frac{1}{2}N^2\sum_{\{\{\vec{k}\}\}} k^2 U(k)\,\xi_{\vec{k}}\,\xi_{-\vec{k}} - \frac{1}{2}N\sum_{\vec{k}} U(k). \qquad (1.9)$$

$\{\{\vec{k}\}\}$ represents the complementary set which contains all the $\vec{k}$ except those belonging to $\{\vec{k}\}$. U(k) is the Fourier coefficient of $U(\vec{x})$:

$$U(\vec{x}) = \sum_{\vec{k}} U(k)\exp(-i\vec{k}\cdot\vec{x}), \qquad (1.10)$$

and $U(k)=V^{-1}4\pi e^2/k^2$ in the present case of Coulomb-interaction (e=electronic charge).

As the next step, we write down the newly-transformed Hamiltonian with collective coordinates (in capital letters) and particle coordinates (in lower case letters). For mere simplicity, we do not put primes on those variables obtained after the transformation (1.7). The Hamiltonian is then expressed as a sum of three terms H_p, H_c and H_{int}.

$$H = H_p(\vec{p},\vec{x}) + H_{int}(\vec{p},\vec{x}; P, Q) + H_c(P, Q); \qquad (1.11)$$

$$H_p = \sum (\vec{p}_j^{\,2}/2m) + U_0(\vec{x}_j), \tag{1.12}$$

$$H_{int} = -\frac{i}{Nm} \sum_{j,\{\vec{k}\}} \frac{(\vec{k}\cdot\vec{p}_j)}{k} \exp(-i\vec{k}\cdot\vec{x}_j) P_{\vec{k}}$$

$$- \frac{1}{2N^2 m} \sum_{\substack{j,\{\vec{k}\},\{\vec{l}\} \\ (\vec{k}+\vec{l}\neq 0)}} P_{\vec{k}} P_{\vec{l}} \frac{(\vec{k}\cdot\vec{l})}{kl} \exp[-i(\vec{k}+\vec{l}\cdot x_j)], \tag{1.13}$$

$$H_c = \frac{1}{2} \sum_{\{\vec{k}\}} [N^2 k^2 U(k) Q_{\vec{k}} Q_{-\vec{k}} + (Nm)^{-1} P_{\vec{k}} P_{-\vec{k}}], \tag{1.14}$$

$$Q_{\vec{k}} = \xi_{\vec{k}}(\vec{x}_1, \cdots, \vec{x}_N) \equiv \frac{1}{Nk} \sum_j \exp(-i\vec{k}\cdot\vec{x}_j) \quad (\text{constraint}) \tag{1.15}$$

As is well known, the electron number density $n(\vec{x})$ is defined as

$$n(\vec{x}) = \sum_j \delta(\vec{x} - \vec{x}_j), \tag{1.16}$$

so that

$$\xi_{\vec{k}} = (Nk)^{-1} n_{\vec{k}}, \tag{1.17}$$

where

$$n_{\vec{k}} = \frac{1}{V} \int n(\vec{x}) \exp(-i\vec{k}\cdot\vec{x})\, d\vec{x}. \tag{1.18}$$

Thus the $\xi_{\vec{k}}$'s are the functions of time t and if we could know the detailed variation of each $\xi_{\vec{k}}$, then the behavior of the $\xi_{\vec{k}}$'s could be readily related to that of the electron density. Furthermore, from the relation

$$\operatorname{div} \vec{E} = 4\pi e \left\{ n(\vec{x}) - \frac{N}{V} \right\}, \quad (e < 0)$$

we have

$$\vec{E}_{\vec{k}} \cdot \vec{E}_{\vec{k}}^{\,*} = (4\pi e N)^2 \, \xi_{\vec{k}} \, \xi_{\vec{k}}^{*} \, , \qquad \text{(1.1}$$

where

$$\vec{E}(\vec{x}) = \sum_{\vec{k}} \vec{E}_{\vec{k}} \exp(-i \vec{k} \cdot \vec{x}) . \qquad \text{(1.}$$

This gives the relation between the electrostatic energy stored in space as a result of electron plasma oscillations and the density fluctuations.

We have so far obtained rigorously the Hamiltonian which constrains the evolution of the collective coordinates. Thus, the Hamiltonian equations for our collective coordinates are as follows:

$$\frac{dQ_{\vec{k}}}{dt} = (Nm)^{-1} P_{-\vec{k}} - \frac{1}{2m} \left[\frac{2i}{N} \sum_{j} \frac{(\vec{k} \cdot \vec{p}_j)}{k} \exp(-i \vec{k} \cdot \vec{x}_j) \right.$$
$$\left. + \, 2 \sum_{\substack{\{\vec{l}\} \\ \vec{k} + \vec{l} \neq 0}} \frac{(\vec{k} \cdot \vec{l})}{N k l} \, |\vec{k} + \vec{l}| \, P_{\vec{l}} \, \xi_{\vec{k} + \vec{l}} \right], \qquad \text{(1.2}$$

$$\frac{dP_{\vec{k}}}{dt} = -N^2 k^2 U(k)\, Q_{-\vec{k}} \,, \qquad (1.22)$$

$$\alpha_{\vec{k}} = \xi_{\vec{k}} \,. \qquad (1.23)$$

The above equations are further simplified by putting

$$Q_{\vec{k}} = \xi_{\vec{k}} \,,$$

$$\omega_p^2 = N k^2 U(k) / m \,, \qquad (1.24)$$

$$P_{-\vec{k}} = N m \omega_p \eta_{-\vec{k}} \qquad (1.25)$$

to yield

$$\frac{d\eta_{\vec{k}}}{dt} = -\omega_p \xi_{\vec{k}} \,, \qquad (1.26)$$

$$\frac{d\xi_{\vec{k}}}{dt} = \omega_p \eta_{\vec{k}} - \sum_{\{\vec{l}\}} \frac{(\vec{k}\cdot\vec{l})}{kl} |\vec{k}+\vec{l}|\, \omega_p \eta_{-\vec{l}} \xi_{\vec{k}+\vec{l}} - i f_{\vec{k}}(t) \,, \qquad (1.27)$$

$$f_{\vec{k}}(t) = (Nm)^{-1} \sum_j \frac{\vec{k}\cdot\vec{p}_j}{k} \exp(-i\vec{k}\cdot\vec{x}_j) \,. \qquad (1.28)$$

Thus we have obtained a set of exact equations from which we start our investigation through the present paper. The second term on the right-hand side of (1.27) represents the effect of nonlinear interaction, i.e. mode-to-mode coupling between waves having distinct wave number vectors. The third term is due to individual particle behavior of the electron plasma; $\vec{p}_j$ is not the whole particle momentum, but the momentum which has been transformed after the aforesaid transformation to extract a collective property of electron plasma.

§1.3 Condition for the predominance of the mode coupling effect

In this chapter, we confine ourselves to the region (k_0, k_c) where the particle aspect is overshadowed by the mode coupling effect. Therefore we shall calculate the approximate value k_c and find on what condition our rather idealized treatment becomes realistic.

The determination of k_c can be carried out by <u>roughly</u> comparing the order of magnitude of the two terms in eq.(1.27):

$$\sum \frac{(\vec{k}\cdot\vec{l})}{k\,l} |\vec{k}+\vec{l}|\, \omega_p\, \eta_{-\vec{l}}\, \xi_{\vec{k}+\vec{l}} \quad \text{and} \quad \frac{1}{Nm}\sum_j \frac{\vec{k}\cdot\vec{p}_j}{k} \exp(-i\vec{k}\cdot\vec{x}_j)$$

Now

$$\left| (Nm)^{-1} \sum_j \frac{\vec{k}\cdot\vec{p}_j}{k} \exp(-i\vec{k}\cdot\vec{x}_j) \right| \sim v k \langle \xi_{\vec{k}}\, \xi^*_{\vec{k}} \rangle^{1/2}$$

(v=thermal velocity of electrons),

$$\left| \sum_{\{\vec{l}\}} \frac{(\vec{k}\cdot\vec{l})}{kl} |\vec{k}+\vec{l}| \omega_p \eta_{-\vec{l}} \xi_{\vec{k}+\vec{l}} \right| \sim \omega_p \sum l \langle \xi_{\vec{l}} \xi^*_{\vec{l}} \rangle$$

(rather overestimated; however, η are of the same order as ξ).

Therefore, in order for the particle aspect to be negligible, it is required that

$$\frac{kv \langle \xi_{\vec{k}} \xi^*_{\vec{k}} \rangle^{1/2}}{\omega_p \sum l \langle \xi_{\vec{l}} \xi^*_{\vec{l}} \rangle} \ll 1 ,$$

i.e.

$$k \ll \frac{\omega_p}{v} \cdot \frac{\sum l \langle \xi_{\vec{l}} \xi^*_{\vec{l}} \rangle}{\langle \xi_{\vec{k}} \xi^*_{\vec{k}} \rangle^{1/2}} . \tag{1.29}$$

Hence we may define k_c as

$$k_c = \left(\frac{\omega_p}{v} \cdot \frac{\sum \langle \xi_{\vec{l}} \xi^*_{\vec{l}} \rangle}{\langle \xi_{\vec{k}} \xi^*_{\vec{k}} \rangle^{1/2}} \right) \cdot k_o . \tag{1.30}$$

Then in case $k_c \gg k_o$, we may neglect the effects due to particle motion in the region (k_o, k_c): as long as we may neglect the effect we have no energy sink in (k_c, k_c) and we may presume that the fluctuation energy is equally distributed among the modes which are in the region (k_o, k_c). To determine the overall slope of the spectral distribution curve, the energy inflow or outflow through the wave number region k_c or k_d must be taken into account and a theory including those particle aspects will be developed in the chapter which follows.

Let us denote the fluctuation energy existing in a mode as $\mathcal{E}$ (per unit volume); then, since

$$\langle \xi_{\vec{k}} \xi_{\vec{k}}^{*} \rangle = \frac{\mathcal{E}}{4\pi e^2 n^2} \tag{1.31}$$

(n=electron number density; we have made use of the fact that $V = k_o^{-3}$, $\mathcal{E} \cdot V = N^2 k^2 U(k) \langle \xi_{\vec{k}} \xi_{\vec{k}}^{*} \rangle$),

we have

$$\sum \langle \xi_{\vec{l}} \xi_{\vec{l}}^{*} \rangle = \frac{\mathcal{E}}{4\pi e^2 n^2} \text{ times the number of degrees of freedom in } (k_o, k_d). \tag{1.32}$$

Thus we have, noticing that the relation $k_d - k_c \ll k_d - k_o$ follows from $k_o \ll k_c$:

$$k_c = \left(\frac{\mathcal{E}_t}{mnv^2} \cdot \frac{k_d^j}{k_o^j} \right)^{1/2} \cdot k_o , \tag{1.33}$$

where

$\mathcal{E}_t$ =total fluctuation energy per unit volume,

$\frac{1}{2} mnv^2$=particle thermal energy per unit volume,

j=3 for 3-dimensional case, and 1 for 1-dimensional case.

Unless the particle dissipation is considered, the slope (which is due to a unidirectional energy flow in the wave number space) cannot be determined; but in either the low temperature limit or the large fluctuation energy limit (i.e. for the case of rather cold plasma) the properties revealed in the following analysis are not far from reality. The type of plasma which submits to the present theory is the plasma which undergoes a sudden disturbance and which takes a certain appreciable lapse of time to effect thermalization.

§1.4 Statistical treatment of the fluctuations

As the Fourier component $\vec{v}_{\vec{k}}$ of a turbulent velocity field is looked upon as a random variable, so we regard the Fourier components $\xi_{\vec{k}}$, $\eta_{\vec{k}}$ as random variables which have a frequency ω_p in the linear approximation, although the phases are indeterminate.

The motive for introducing the concept of randomness is quite analogous to that of hydrodynamical turbulence;[2] it follows from the randomness of fluctuations that the Fourier components are also random. To see the average behavior of the fluctuations, we may make averaging operations over the random variables by enclosing them with brackets. e.g. $\langle ** \rangle$. The average is a spatial average inasmuch as the fluctuations are homogeneous in space. Or, what amounts to the same thing, the average is equivalent to a time average because the phenomena of the present study are a random process stationary in time.

In what follows, we shall state that the statistical distribution of the fluctuation can be assumed to be quasi-normal. As a matter of fact, we shall verify in what follows that the quasi-normality is a very probable type of distribution and that such types of distribution are the approximate solutions to the stationary cases of our interest.

The clues to the verification are found in Hopf's paper.[3] See for more details the account of his theory in reference 3. He showed, by tracing the behavior of the characteristic function, that the phase distribution of the fluid velocity can be actually normal in the limiting case of zero viscosity and that the energy equipartition is developed in the wave number space in such a turbulent flow.

Hopf states that, in the n-dimensional $\underline{u}$-space, $\underline{u}=(\underline{u}_1, \cdots, \underline{u}_n)$, denoted also by Ω , the differential law of the phase motion becomes an ordinary differential system:--

$$\frac{du_\nu}{dt} = \alpha_\nu(u) = \alpha_\nu(u_1, \cdots, u_n), \quad \nu = 1, \cdots, n. \tag{1.34}$$

The functions α are assumed to be polynomials. In the hydrodynamical case the right-hand sides of the differential equations of phase motion are functionals of second degree $u = u(\vec{x})$.. The characteristic function Φ is defined as

$$\left.\begin{aligned} \Phi(y,t) &= \overline{\exp\{i(y,u)\}} = \int_\Omega \exp\{i(y,u)\} F^t(du) \\ (y,u) &= \sum y_\nu u_\nu \ ; \end{aligned}\right\} \tag{1.35}$$

$P^t(du)$ denotes the "differential" probability phase distribution after a lapse of time t from the initial instance. In general it changes with time t. (If, however, the underlying phase distribution is stationary, then the average of any phase function stays constant in time --- this is the present case.) The characteristic function Φ of any phase distribution $P^t(A)$ satisfies the linear partial differential equation

$$\frac{\partial \Phi}{\partial t} = i \sum y_\nu Q_\nu \left(\frac{\partial}{i\partial y_1}, \cdots, \frac{\partial}{i\partial y_n} \right) \Phi , \qquad (1.36)$$

in which the expressions Q_ν on the right are to be interpreted as symbolic differential operators.

Now we shall apply the above relations to the present problem and find under what conditions the phases $\xi_{\vec{k}}$ and $\eta_{\vec{k}}$ permit a normal distribution which is also a stationary solution $\partial\Phi/\partial t = 0$.

First, we define as follows:

$$\left.\begin{aligned} [y, \xi'] &= \sum_{\{\vec{k}\}} y_{\vec{k}} \xi'_{\vec{k}} , \\ [z, \eta'] &= \sum_{\{\vec{k}\}} z_{\vec{k}} \eta'_{\vec{k}} ; \\ y^*_{\vec{k}} = y_{-\vec{k}} &, \quad z^*_{\vec{k}} = z_{-\vec{k}} , \end{aligned}\right\} \qquad (1.37)$$

$$\xi'_{\vec{k}} = k_c \xi^*_{\vec{k}} , \qquad \eta'_{\vec{k}} = k_c \eta^*_{\vec{k}} .$$

The characteristic function is:

$$\Phi(y, z; t) = E\left[\exp i([y, \xi'] + [z, \eta'])\right]. \tag{1.38}$$

where E represents the expectation of the value enclosed within the square brackets.

As $\xi'_{\vec{k}}$ and $\eta'_{\vec{k}}$ satisfy

$$\left.\begin{aligned} \frac{d\eta'_{\vec{k}}}{dt} &= -\omega_p \xi'_{\vec{k}} , \\ \frac{d\xi'_{\vec{k}}}{dt} &= \omega_p \eta'_{\vec{k}} - \sum_{\{\vec{l}\}} \frac{(\vec{k}\cdot\vec{l})}{k l k_c} |\vec{k}+\vec{l}| \eta'_{-\vec{l}} \xi'_{\vec{k}+\vec{l}} , \end{aligned}\right\} \tag{1.39}$$

we have from (1.36)

$$\frac{1}{\omega_p}\frac{\partial \Phi}{\partial t} = \sum_{\{\vec{k}\}} y_{\vec{k}} \frac{\partial \Phi}{\partial z_{\vec{k}}} + i \sum_{\{\vec{k}\}}\sum_{\{\vec{l}\}} \frac{(\vec{k}\cdot\vec{l})|\vec{k}+\vec{l}|}{k l k_c} y_{\vec{k}} \frac{\partial^2 \Phi}{\partial y_{\vec{k}+\vec{l}}\, \partial z_{-\vec{l}}} - \sum_{\{\vec{k}\}} z_{\vec{k}} \frac{\partial \Phi}{\partial y_{\vec{k}}} . \tag{1.40}$$

Thus if we could assume that

$$\Phi = \exp\left\{-\kappa^2 \sum_{\{\vec{k}\}} (y_{\vec{k}} y^*_{\vec{k}} + z_{\vec{k}} z^*_{\vec{k}})\right\} \tag{1.41}$$

where $\varkappa^2$ is a dimensionless constant, then we would have

$$\frac{1}{\omega_p}\frac{\partial\Phi}{\partial t} = i\frac{\cdot}{\tau}\varkappa^4 \exp\left\{-\varkappa^2\sum\left(y_{\vec{k}}y^*_{\vec{k}} + z_{\vec{k}}z^*_{\vec{k}}\right)\right\}\cdot\sum\sum\frac{(\vec{k}\cdot\vec{l})|\vec{k}+\vec{l}|}{k\,l\,k_c}y_{\vec{k}}z_{\vec{l}}y^*_{\vec{k}+\vec{l}}\ldots \qquad (1.42)$$

Denoting the absolute value of the right-hand side of the above equation as δ, we assert that the sufficient conditions for δ to be negligible are as follows:

(1) $$|y_{\vec{k}}| \text{ or } |z_{\vec{k}}| \rightarrow \infty ; \qquad (1.43)$$

(2) $$|y_{\vec{k}}| \text{ and } |z_{\vec{k}}| \rightarrow 0 . \qquad (1.44)$$

Therefore, to ensure that δ is negligible for medium values of $|y_{\vec{k}}|$ and $|z_{\vec{k}}|$, another sufficient condition is found to be necessary: viz.,

(3) $$\varkappa^4 \ll 1 . \qquad (1.45)$$

Therefore, when the condition (3) is satisfied, the characteristic function defined in (1.41), which corresponds to a normal phase distribution, is one approximate stationary solution to Eq.(1.40).

To see the physical meaning of the constant $\varkappa^2$ we expand the characteristic function Φ in a Taylor series:

$$\Phi = \Phi^0 + \Phi^1 + \Phi^2 + \cdots , \qquad (1.46)$$

then

$$\Phi^n = \sum \cdots \sum G(\vec{k}^1, \cdots, \vec{k}^n)\, y_{\vec{k}^1} \cdots y_{\vec{k}^i} \cdots z_{\vec{k}^n} \,, \qquad (1.47)$$

The coeffients G are (cf. (1.38))

$$\frac{1}{n!} \frac{\partial^n \Phi}{\partial y_{\vec{k}^1} \cdots \partial y_{\vec{k}^i} \cdots \partial z_{\vec{k}^n}} = \frac{i^n}{n!} \langle \xi_{\vec{k}^1} \cdots \xi_{\vec{k}^i} \cdots \eta_{\vec{k}^n} \rangle. \qquad (1.48)$$

We then have (cf. (1.57)):

$$\Phi^1 = 0 \,, \qquad (1.49)$$

$$\begin{aligned}\Phi^2 &= -\frac{1}{2} \sum\sum \Big[\langle \xi'_{\vec{k}^1} \xi'_{\vec{k}^2} \rangle y_{\vec{k}^1} y_{\vec{k}^2} + \langle \xi'_{\vec{k}^1} \eta'_{\vec{k}^2} \rangle y_{\vec{k}^1} z_{\vec{k}^2} \\ &\qquad + \langle \eta'_{\vec{k}^1} \eta'_{\vec{k}^2} \rangle z_{\vec{k}^1} z_{\vec{k}^2} \Big] \\ &= -\sum_{\{\vec{k}\}} \Big[\langle \xi'_{\vec{k}} \xi^{*\prime}_{\vec{k}} \rangle y_{\vec{k}} y^*_{\vec{k}} + \langle \eta'_{\vec{k}} \eta'^*_{\vec{k}} \rangle z_{\vec{k}} z^*_{\vec{k}} \Big]. \end{aligned} \qquad (1.50)$$

As we know for the present case

$$\Phi^2 = -\varkappa^2 \sum_{\{\vec{k}\}} (y_{\vec{k}} y^*_{\vec{k}} + z_{\vec{k}} z^*_{\vec{k}}) \,, \qquad (1.51)$$

therefore,

$$\varkappa^2 = \langle \xi'_{\vec{k}} \xi^{*\prime}_{\vec{k}} \rangle = \langle \eta'_{\vec{k}} \eta^{*\prime}_{\vec{k}} \rangle = \frac{1}{2} \Big(\langle \xi'_{\vec{k}} \xi^{*\prime}_{\vec{k}} \rangle + \langle \eta'_{\vec{k}} \eta^{*\prime}_{\vec{k}} \rangle \Big). \qquad (1.52)$$

In other words, since $\langle \xi'_{\vec{k}} \xi'^{*}_{\vec{k}} \rangle$ is proportional to the fluctuation energy allotted to a longitudinal mode with wave vector $\vec{k}$ (cf. Eq.(1.31)), Eq.(1.52) says that the energy is equipartitioned among various longitudinal modes.

Hence the sufficient condition for the right-hand side of (1.40) to vanish for any y and z is, from (1.45):

$$\kappa^4 = k_c^4 \left(\frac{\varepsilon}{4\pi e^2 n^2} \right)^2 \ll 1 \tag{1.53}$$

or, from (1.33):

$$\left(\frac{\varepsilon_t}{m n v^2} \cdot \frac{\varepsilon_t k_0^{-3}}{(4\pi e^2 k_0)(n k_0^{-3})^2} \right)^2 \ll 1, \tag{1.54}$$

where

$\varepsilon_t k_0^{-3}$ =total fluctuation energy,

$4\pi e^2 k_0$ =potential energy between a pair of electrons spaced $2\pi k_0^{-1}$ apart,

$n k_0^{-3}$ =total number of electrons.

Thus when this condition of long wavelength (cf.(1.53))or high density (cf.(1.54)) is met, the normal distribution is a stable stationary solution approximately, because we then have

$$\frac{1}{\omega_p} \cdot \frac{\partial \Phi}{\partial t} = O(\kappa^4). \tag{1.55}$$

We have roughly examined the condition for the validity of the assumption of the quasi-normality of the distribution $\xi_{\vec{k}}$, $\eta_{\vec{k}}$. From this, we can expand approximately, as in the case of a pure normal distribution:

$$\langle XYZW \rangle = \langle XY \rangle \langle ZW \rangle + \langle XZ \rangle \langle YW \rangle + \langle XW \rangle \langle YZ \rangle. \qquad (1.56)$$

Here X, Y, Z and W are either of ξ or η. We make use of the above approximate relation between the fourth order correlation and the second correlations, in order that we can derive a closed system of equations. The expansion is possible for any large amplitude phenomena so long as the condition (1.53) or (1.54) is satisfied.

Now we should mention another important property which permits simplifications. Generally, when any random variable, which is a space-time function, is analyzed into Fourier components by Fourier spatial transform, then, as long as the original space-time function can be regarded as having spatial statistical homogeneity, the Fourier coefficients (functions of time only) are also random variables and they have orthogonal properties with respect to wave number vectors:[4)]

$$\langle \xi_{\vec{k}}(t)\, \xi^*_{\vec{k}'}(t) \rangle \begin{cases} = 0, & \vec{k} \neq \vec{k}'; \\ \neq 0, & \vec{k} = \vec{k}'. \end{cases} \qquad (1.57)$$

The above relation should not be confused with the so-called random phase approximation which is often assumed in dealing with the particle aspect of many-particle systems.

§1.5 Calculations

We now confine ourselves to the long wavelength region (k_o, k_c) where the mode-coupling effects prdominate as discussed in §1.3, and perform a Laplace transformation on the following equations whose derivation is in the Appendix 1. (Simplified notations such as $\xi_{\vec{k}}(t) = \xi_{\vec{k}}$, $\xi_{\vec{k}}(0) = \xi^{o}_{\vec{k}}$, etc., are used.)

$$\frac{d^2}{dt^2} \langle \xi_{\vec{k}} \xi^{*o}_{\vec{k}} \rangle = -\omega_p^2 \langle \xi_{\vec{k}} \xi^{*o}_{\vec{k}} \rangle - \sum_{\{\vec{l}\}} \frac{(\vec{k}\cdot\vec{l})}{kl} |\vec{k}+\vec{l}| \, \omega_p \frac{d}{dt} \langle \eta_{-\vec{l}} \, \xi_{\vec{k}+\vec{l}} \, \xi^{*o}_{\vec{k}} \rangle \quad \ldots\ldots \tag{1.58}$$

$$\frac{d}{dt} \begin{pmatrix} \langle \eta_{-\vec{l}} \, \xi_{\vec{k}+\vec{l}} \, \xi^{*o}_{\vec{k}} \rangle \\ \langle \xi_{-\vec{l}} \, \xi_{\vec{k}+\vec{l}} \, \xi^{*o}_{\vec{k}} \rangle \\ \langle \xi_{-\vec{l}} \, \eta_{\vec{k}+\vec{l}} \, \xi^{*o}_{\vec{k}} \rangle \\ \langle \eta_{-\vec{l}} \, \eta_{\vec{k}+\vec{l}} \, \xi^{*o}_{\vec{k}} \rangle \end{pmatrix} = \begin{pmatrix} 0 & -\omega_p & 0 & \omega_p \\ \omega_p & 0 & \omega_p & 0 \\ 0 & -\omega_p & 0 & \omega_p \\ -\omega_p & 0 & -\omega_p & 0 \end{pmatrix} \times$$

$$\times \begin{pmatrix} \langle \eta_{-\vec{l}}\, \xi_{\vec{k}+\vec{l}}\, \xi^{*0}_{\vec{k}} \rangle \\ \langle \xi_{-\vec{l}}\, \xi_{\vec{k}+\vec{l}}\, \xi^{*0}_{\vec{k}} \rangle \\ \langle \xi_{-\vec{l}}\, \eta_{\vec{k}+\vec{l}}\, \xi^{*0}_{\vec{k}} \rangle \\ \langle \eta_{-\vec{l}}\, \eta_{\vec{k}+\vec{l}}\, \xi^{*0}_{\vec{k}} \rangle \end{pmatrix}$$

$$+ \ \omega_p \begin{pmatrix} \dfrac{\vec{k}+\vec{l}\cdot\vec{l}}{|\vec{k}+\vec{l}|\,l}\, k \langle \eta_{-\vec{l}}\, \eta_{\vec{l}} \rangle \langle \xi_{\vec{k}}\, \xi^{*0}_{\vec{k}} \rangle \\ \left\{ \dfrac{\vec{k}+\vec{l}\cdot\vec{k}}{|\vec{k}+\vec{l}|\,k}\, l \langle \xi_{\vec{l}}\, \xi^{*}_{\vec{l}} \rangle - \dfrac{\vec{k}\cdot\vec{l}}{k\,l}\, |\vec{k}+\vec{l}| \langle \xi_{\vec{k}+\vec{l}}\, \xi^{*}_{\vec{k}+\vec{l}} \rangle \right\} \\ \dfrac{\vec{k}+\vec{l}\cdot\vec{l}}{|\vec{k}+\vec{l}|\,l}\, k \langle \eta_{-\vec{k}-\vec{l}}\, \eta_{\vec{k}+\vec{l}} \rangle \langle \xi_{\vec{k}}\, \xi^{*0}_{\vec{k}} \rangle \\ 0 \end{pmatrix}, \qquad (1.59)$$

$$\frac{d}{dt} \langle \eta_{\vec{k}}\, \xi^{*0}_{\vec{k}} \rangle = -\omega_p \langle \xi_{\vec{k}}\, \xi^{*0}_{\vec{k}} \rangle. \qquad (1.60)$$

We assumed as initial conditions:

$$\left\langle \left(\frac{d}{dt} \xi_{\vec{k}} \right)^0 \xi^{*0}_{\vec{k}} \right\rangle = 0, \qquad (1.61)$$

$$\begin{pmatrix} \langle \eta^0_{-\vec{l}}\, \xi^0_{\vec{k}+\vec{l}}\, \xi^{*0}_{\vec{k}} \rangle \\ \langle \xi^0_{-\vec{l}}\, \eta^0_{\vec{k}+\vec{l}}\, \xi^{*0}_{\vec{k}} \rangle \\ \langle \xi^0_{-\vec{l}}\, \eta^0_{\vec{k}+\vec{l}}\, \xi^{*0}_{\vec{k}+\vec{l}} \rangle \\ \langle \eta^0_{-\vec{l}}\, \eta^0_{\vec{k}+\vec{l}}\, \xi^{*0}_{\vec{k}} \rangle \end{pmatrix} = 0, \qquad (1.62)$$

then the Laplace transform of $\langle \xi_{\vec{k}} \xi_{\vec{k}}^{*o} \rangle$ is:

$$L\left[\langle \xi_{\vec{k}} \xi_{\vec{k}}^{*o} \rangle\right] = +s \langle \xi_{\vec{k}} \xi_{\vec{k}}^{*} \rangle \Big/ \Big[s^2 + \omega_p^2 + \frac{\omega_p^2}{s^2+4\omega_p^2} \sum_{\{\vec{l}\}} \frac{(\vec{k}\cdot\vec{l})}{kl}$$

$$\times |\vec{k}+\vec{l}| \Big\{ (s^2+2\omega_p^2) \frac{\vec{k}+\vec{l}\cdot\vec{l}}{|\vec{k}+\vec{l}|\, l} k \langle \eta_{-\vec{l}} \eta_{\vec{l}} \rangle$$

$$+ \omega_p^2 \left(\frac{\vec{k}+\vec{l}\cdot\vec{k}}{|\vec{k}+\vec{l}|\, k} l \langle \xi_{\vec{l}} \xi_{\vec{l}}^{*} \rangle - \frac{(\vec{k}\cdot\vec{l})}{kl} |\vec{k}+\vec{l}| \langle \xi_{\vec{k}+\vec{l}} \xi_{\vec{k}} \right.$$

$$- 2\omega_p^2 \frac{\vec{k}+\vec{l}\cdot\vec{l}}{|\vec{k}+\vec{l}|\, l} k \langle \eta_{-\vec{k}-\vec{l}} \eta_{\vec{k}+\vec{l}} \rangle \Big\} \Big]. \tag{1.63}$$

In the above, we have made use of the fact that, in the present case, products such as $\langle \xi_{\vec{k}} \xi_{\vec{k}}^{*} \rangle$ do not depend on time.

We write

$$s = -i\omega - \delta \qquad (\omega, \delta : \text{real numbers}) \tag{1.64}$$

and equate the denominator of Eq.(1.63) to zero. Then we have two equations, one from the real part, the other from the imaginary part, from which we know that two types of solutions exist:

$$(a) \qquad \delta^2(\vec{k}) = 0, \tag{1.65}$$

$$\omega^2(\vec{k}) = \frac{1}{2}\omega_p^2 \Big[\Big\{ 5 + \sum_{\{\vec{l}\}} \frac{(\vec{k}\cdot\vec{l})(\vec{k}+\vec{l}\cdot\vec{l})}{l^2} \langle \eta_{-\vec{l}}\, \eta_{\vec{l}} \rangle$$

$$\pm \Big[\Big\{ 5 + \sum \frac{(\vec{k}\cdot\vec{l})(\vec{k}+\vec{l}\cdot\vec{l})}{l^2} \langle \eta_{-\vec{l}}\, \eta_{\vec{l}} \rangle \Big\}^2 - 4 \Big[4 + \sum \frac{(\vec{k}\cdot\vec{l})}{kl} \cdot$$

$$\times |\vec{k}+\vec{l}| \Big\{ \frac{\vec{k}+\vec{l}\cdot\vec{l}}{|\vec{k}+\vec{l}|\, l} k \cdot 2 \left(\langle \eta_{-\vec{l}}\, \eta_{\vec{l}} \rangle - \langle \eta_{-\vec{k}-\vec{l}}\, \eta_{\vec{k}+\vec{l}} \rangle \right)$$

$$+ \frac{\vec{k}+\vec{l}\cdot\vec{k}}{|\vec{k}+\vec{l}|\, k} l \langle \xi_{\vec{l}}\, \xi^*_{\vec{l}} \rangle - \frac{(\vec{k}\cdot\vec{l})}{kl} |\vec{k}+\vec{l}| \langle \xi_{\vec{k}+\vec{l}}\, \xi^*_{\vec{k}+\vec{l}} \rangle \Big\} \Big] \Big]^{1/2} \Big]; \tag{1.66}$$

(b) $$\gamma^2(\vec{k}) = \omega_p^2 \Big[\frac{1}{2} \Big[4 + \sum_{\{\vec{l}\}} \cdots \Big]^{1/2} - \frac{1}{4} \Big\{ 5 + \sum \cdots \Big\} \Big], \tag{1.67}$$

$$\omega^2(\vec{k}) = \omega_p^2 \Big[\frac{1}{4} \Big\{ 5 + \sum \cdots \Big\} + \frac{1}{2} \Big[4 + \sum \cdots \Big]^{1/2} \Big]. \tag{1.68}$$

As can be seen from (1.58), if initially the turbulent state of the fluctuating electric field is one-dimensional, then the state will ever remain one-dimensional. since there is no coupling between waves with wave number vector orthogonal to each other. For such a one-dimensional case, taking into account the statistical symmetry, we then simplify as follows:

$$\Big[4 + \sum \cdots \Big] = 4 + 4k^2 \sum \langle \eta_{-\vec{l}}\, \eta_{\vec{l}} \rangle + \sum l^2 \langle \xi_{\vec{l}}\, \xi^*_{\vec{l}} \rangle$$

$$- \sum (\vec{k}+\vec{l})^2 \langle \xi_{\vec{k}+\vec{l}} \, \xi^*_{\vec{k}+\vec{l}} \rangle - 2k^2 \sum \langle \eta_{\vec{k}+\vec{l}} \, \eta_{-\vec{k}-\vec{l}} \rangle ,$$

(1. 69)

$$\{ 5 + \sum \cdots \} = 5 + k^2 \sum \langle \eta_{-\vec{l}} \, \eta_{\vec{l}} \rangle .$$

(1. 70)

For $\vec{k}$ whose absolute magnitude is small enough, it can be easily shown that $\gamma^2(\vec{k})$ of (1.67) is negative, hence $\gamma(\vec{k})$ is not real. Also for $\vec{k}$ whose magnitude is large enough, making use of another assumption plausible for the steady state $\langle \eta_{-\vec{l}} \eta_{\vec{l}} \rangle \sim \langle \xi_{\vec{l}} \xi^*_{\vec{l}} \rangle$(cf. Eq.(1.52)), it can readily be proved that $\gamma^2(\vec{k})$ is of negative value. Thus, in short, the solution of (b) is not a physical solution, since it contradicts the conditi.... that $\gamma(\vec{k})$ be real.

Let us scrutinize the solution (a). Of the two branches, we call one the high-frequency mode, the other the low- requency mode, corresponding to +/- sign in (1.66) respectively. The fact that $\gamma(\vec{k})=0$ reflects the situation where the energy interchange between the modes occurs reversibly and where the energy flow in the wave number space is not unidirectional statistically. In other words, we have not considered the particle aspect; therefore, in the steady state, even if one mode transfers its energy to other modes, the latter gives back the energy to the former, thus giving rise to no net damping.

§1.6 Rate of electric field fluctuation

For long wavelengths (i.e. for k, $k_0 < k < k_c$), the obtaine ω-versus-k relations assume much simpler form. As we are dealing with a steady state, using the relation(1.52), we have

Low-frequency mode:

$$\omega^2(\vec{k}) = \frac{1}{2}\omega_p^2 \left[5 + k^2 \Sigma - \sqrt{k^4 \Sigma^2 + 2k^2 \Sigma + 9} \right], \tag{1.71}$$

High-frequency mode:

$$\omega^2(\vec{k}) = \frac{1}{2}\omega_p^2 \left[5 + k^2 \Sigma + \sqrt{k^4 \Sigma^2 + 2k^2 \Sigma + 9} \right], \tag{1.72}$$

where

$$\Sigma \equiv \sum_{\{\vec{\ell}\}} \langle \xi_{\vec{\ell}} \xi^*_{\vec{\ell}} \rangle = \frac{\varepsilon_t}{4\pi e^2 n^2}. \tag{1.73}$$

We have plotted the above relation in Fig. 1.1. We then split the right-hand side of Eq.(1.63) into partial fractions in order to determine their relative weights. The ratio of the amplitude of those two modes is also shown in Fig. 1.1. Thus we see that when either k is small or the fluctuation energy is not so great, then the predominant mode is the low-frequency mode. This is the continuation of the linear approximation. In case the k or the fluctuation energy is quite large, the high frequency predominates.

We have so far tried to answer the question: What is the

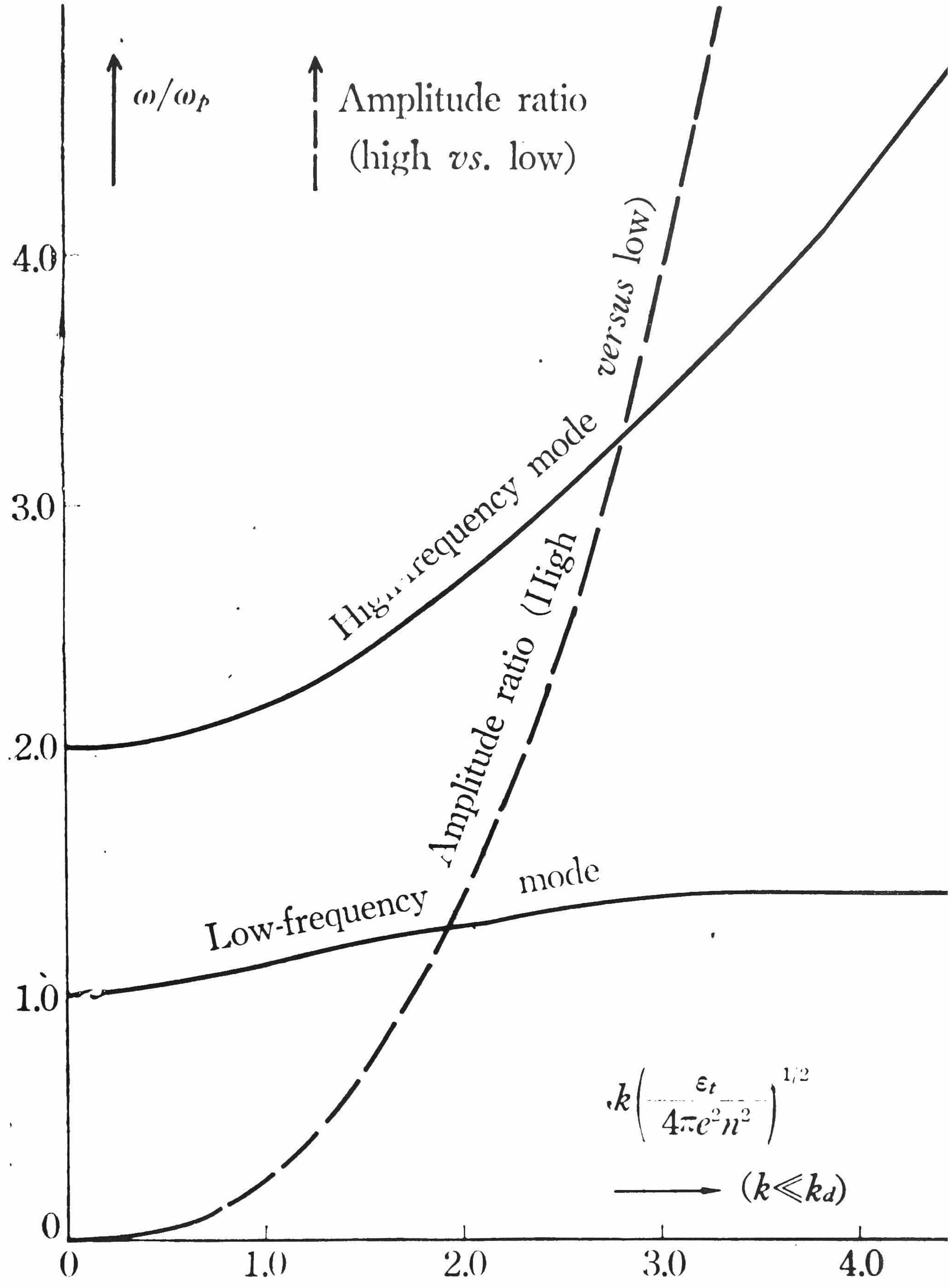

Fig. 1.1 Mean rate of electric field fluctuations ($2\pi/k$ is the scale of measurement in space).

mean rate of electric field fluctuations when we put two points of probes inside a well-agitated plasma? The spacing between the two probes is of the order of $2\pi/\underline{k}$ and we measure the fluctuating potential difference between the two thin conductors. If we can prescribe the total amount of the fluctuation energy, then we can conclude theoretically that the mean rate of electric field (or density) fluctuations with wavelength $2\pi/\underline{k}$ is that which can be read from the curve in Fig. 1.1.

We have to await the time when we will experimentally study the fine structure of the plasma. But at least we can say that to the extent that we can assume the validity of the quasi-normality of the phase distribution, the phase fluctuation will occur at the rate we have derived. The merit of the present theory is that the method is not that of a perturbation technique. It should be remembered that the (quasi-) normality is not necessarily equivalent to linearization.

The electric field fluctuations or the electron density fluctuations observable at one point of space generally have various frequency components. They can be known from the $\underline{\omega}$ -versus- $\underline{k}$ relation by varying the value of $\underline{k}$ within the range compatible to the relevant physical conditions.

§1.7 Lifetime of an irregularity in density

As has been stated in the first section of this chapter, the picture of the turbulent electric field fluctuations is one in which the longitudinal density pattern is analyzed into waves with various wave numbers. Thus if we mention a certain wave number, $\vec{k}$, then we are referring to the waves with various phases in space, which have been produced by nonlinear interactions between waves with longer (or possibly shorter) wavelengths. When we observe a longitudinal density pattern, whose mean wavelength is $2\pi/\underline{k}$, then we can say that waves with wavelength equal to or less than $2\pi/\underline{k}$ have been in phase spatially; but once this coincidence has been established, then each component of the waves contributing to the formation of the irregularity begins to oscillate with various and differing frequencies as shown in the preceding section. After a certain lapse of time --- we define this as the lifetime of the irregularity --- the irregularity almost completely disappears in the phase-mixing process.

So let us attempt to evaluate the lifetime of an irregularity from Eq.(1.71) and (1.72).

As can easily be ascertained, we know that

$$\omega = \omega_p \left\{ 1 + \frac{1}{6} k^2 \Sigma - \frac{19}{216} k^4 \Sigma^2 + O(k^6 \Sigma^3) \right\}, \quad \left(k^2 \Sigma \ll 1 \ ; \ \text{Cf. } (1.71) \right) \tag{1.74}$$

and

$$\left.\begin{aligned} \omega &= \omega_p k_1 \sqrt{\Sigma}\left\{1+\frac{3}{2}\left(k^2\Sigma\right)^{-1}+O\left(\left(k^2\Sigma\right)^{-2}\right)\right\}, \\ &\left(k^2\Sigma \gg 1\ ;\ \text{Cf. } (1.72)\right) \end{aligned}\right\} \qquad (1.75)$$

We may define the lifetime of an irregularity with wavelength of the order of $2\pi/k_1$ as the time interval after which the inequality

$$\left| \sum_{k_0<k<k_1} \langle \xi_{\vec{k}}\, \xi_{\vec{k}}^{*0} \rangle \Big/ \sum_{k_0<k<k_1} \langle \xi_{\vec{k}}\, \xi_{\vec{k}}^{*} \rangle \right| \ll 1. \qquad (1.76)$$

is satisfied. As we are dealing with a one-dimensinal case, the above inequality can be written as

$$\left| \int_{k_0}^{k_1} \langle \xi_{\vec{k}}\, \xi_{\vec{k}}^{*0} \rangle\, dk \right| \Big/ \int_{k_0}^{k_1} \langle \xi_{\vec{k}}\, \xi_{\vec{k}}^{*} \rangle\, dk \ll 1. \qquad (1.77)$$

Moreover, since the case of our interest is a steady state where $\langle \xi_{\vec{k}}\, \xi_{\vec{k}}^{*} \rangle$ is a constant irrelevant to k, (1.77) reduces to

$$\frac{1}{k_1-k_0}\left| \int_{k_0}^{k_1} \exp\{i\omega(\vec{k})t\}\, dk \right| \ll 1. \qquad (1.78)$$

First, for the case $k^2\Sigma \ll 1$, we have

$$\int_{k_0}^{k_1} \exp\{i\omega(\vec{k})t\}\,dk \cong \exp\{i\omega_p t\}\int_{k_0}^{k_1} \exp\{i\omega_p t\cdot\frac{1}{6}k^2\Sigma\}\,dk,$$

where

$$\Sigma = \sum_{\{\vec{k}\}} \langle \xi_{\vec{k}}\, \xi_{\vec{k}}^{*} \rangle$$

has already been defined in (1.73).

Therefore, denoting the left-hand side of (1.78) as "R", we have

$$R = \frac{1}{k_1 - k_0}\cdot\frac{1}{\sqrt{(1/6)\omega_p t\Sigma}}\left|\int_{k_0\sqrt{(1/6)\omega_p t\Sigma}}^{k_1\sqrt{(1/6)\omega_p t\Sigma}} \exp\{iu^2\}\,du\right|. \tag{1.79}$$

We have to find the value of t which makes R appreciably smaller than unity. The complex value of the definite integral in (1.78) can be obtained from the diagram of the well-known Cornu spiral. If $k_1 \gg k_0$, then

$$R \sim \frac{1}{k_1\sqrt{(1/6)\omega_p t\Sigma}}\left|\int_{0}^{k_1\sqrt{(1/6)\omega_p t\Sigma}} \exp\{iu^2\}\,du\right|;$$

and the value of $\left|\int_0^x e^{iu^2}\,du\right|$ converges to $\sqrt{0.5^2+0.5^2}$,

hence the behavior of R is determined by $1/\{k_1\sqrt{(1/6)\omega_p t\sum}\}$.
If $k_1 > k_0$ (k_1 not much larger than k_0), the value of $\left|\int\right|$ in (1.79) tends to zero; on the other hand the value $k_1\sqrt{(1/6)\cdot\omega_p t\sum}$ which makes $\left|\int\right|$ almost vanish is more than of the order of ten, so that when $k_1\sqrt{(1/6)\omega_p t\sum}$ has increased to ten with increasing t, the factor $1/\{k_1\sqrt{\frac{1}{6}\omega_p t\sum}\}$ is again predominant. Thus in general, for the case $k_1^2\sum \ll 1$, the amplitude of the irregularity diminishes approximately at the rate of $1/\{k_1\sqrt{\frac{1}{6}\omega_p t\sum}\}$ ... Therefore, we can conclude that for this case the lifetime is $1/\{k_1^2\cdot\frac{1}{6}\cdot\omega_p\sum\}$, though the rate of decay is rather slower than that conventionally defined (the amplitude of the irregularity varies as $1/\sqrt{t}$).

For the other case $k_1^2\sum \gg 1$ and $k_0^2\sum \gg 1$.

$$\left|\int_{k_0}^{k_1} exp\{i\omega(\vec{k})t\}\,dk\right| = \frac{1}{\omega_p\sqrt{\sum}\,t}\left|exp\{i\omega_p\sqrt{\sum}\,tk_1\} - exp\{i\omega_p\sqrt{\sum}\,tk_0\}\right|,$$

so that

$$R = \frac{1}{(k_1-k_0)\omega_p\sqrt{\sum}\,t}\left|exp\{i\omega_p\sqrt{\sum}\,tk_1\} - exp\{i\omega_p\sqrt{\sum}\,tk_0\}\right|. \tag{1.80}$$

Therefore the lifetime of the irregularity with wavelength $2\pi/k_1$ is about $1/\{(k_1-k_0)\omega_p\sqrt{\sum}\}$. As we have assumed $k_1^2\sum \gg 1$,

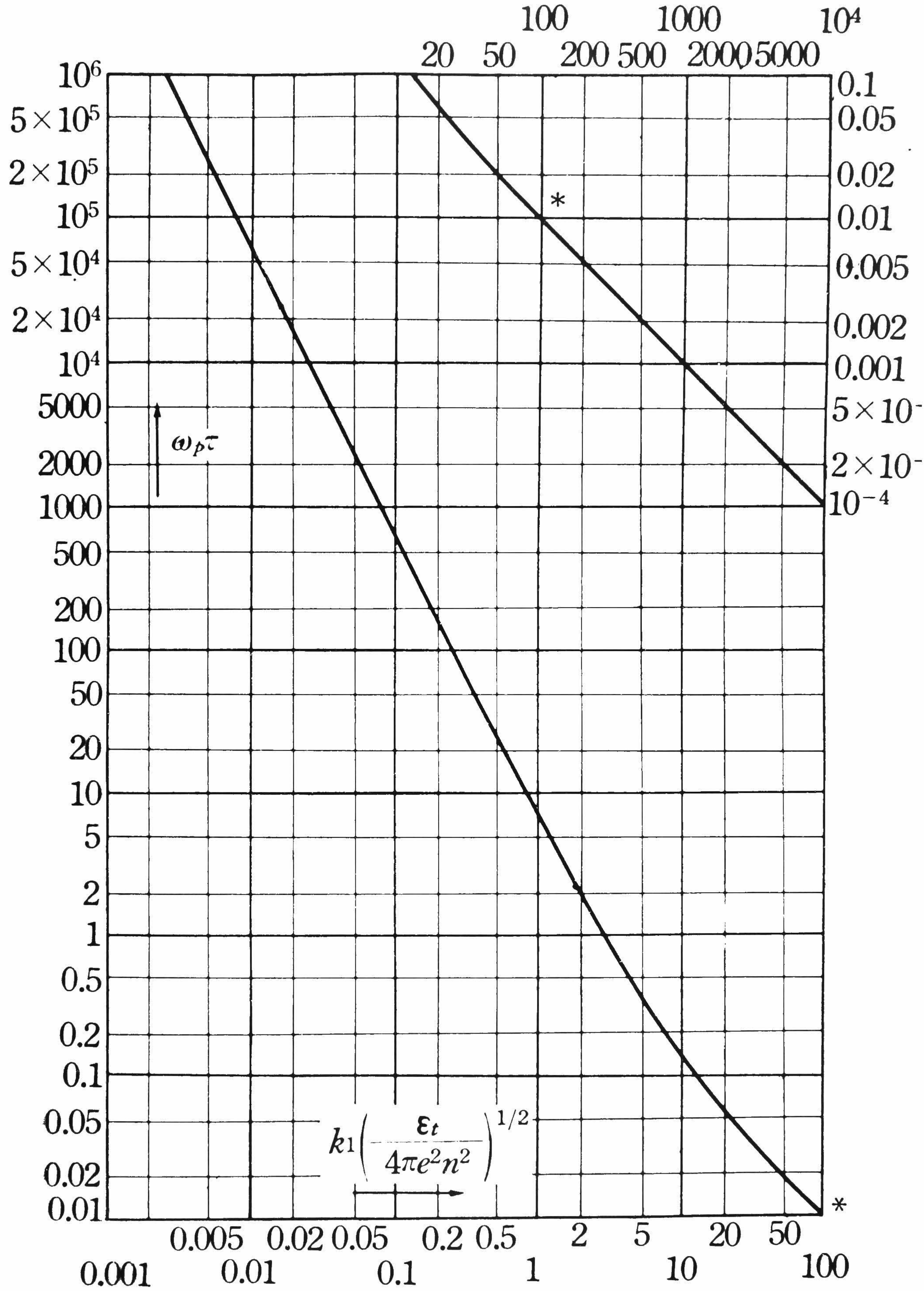

Fig. 1.2 Lifetime τ of an irregularity

the lifetime is much shorter than that corresponding to the period of plasma oscillation except for the case in which k_1 is slightly larger than k_0.

When there is not much difference between k_1 and k_0, the situation is critical and the lifetime can be longer than the period of plasma oscillations.

The approximate values of the lifetime so far obtained are:

$$\tau = \begin{cases} \dfrac{24\pi e^2 n^2}{k_1^2 \varepsilon_t} \cdot \dfrac{1}{\omega_p}, & \left(k_1^2 \dfrac{\varepsilon_t}{4\pi e^2 n^2} \ll 1 \right); \quad (1.81) \\[2ex] \dfrac{\sqrt{4\pi e^2 n^2}}{(k_1 - k_0)\sqrt{\varepsilon_t}} \cdot \dfrac{1}{\omega_p}, & \left(k_0^2 \dfrac{\varepsilon_t}{4\pi e^2 n^2} \gg 1 \right). \quad (1.82) \end{cases}$$

These values are plotted against $k_1\sqrt{\varepsilon_t/4\pi e^2 n^2}$ in Fig.1.2. (k_0 is regarded as small enough for the high-frequency mode).

§1.8 Concluding remarks

In this chapter, we have drawn two conclusions: One is about the mean rate of fluctuation of the electric field strength in an agitated electron plasma and the other about the lifetime (or, conversely, the growth time) of an irregularity in density. In the light of the present status of the theoretical investigation concerning the nonlinear aspects of the plasma, the results may be noteworthy since the technique utilized in the present theory is non-perturbational, a combination being made most effectively between the method of redundant variables and the stability of the Gaussian distribution.

We have so far idealized the plasma in such a way that the discussions may be self-contained within the scope of the collective behavior only. In the situations where the conditions for such treatment to be valid break down, the interrelationship between the particle aspect and the collective behavior looms up. The unified treatment for both of these two aspects will be developed in chapters which follow.

REFERENCES (Chapter 1)

1) D. ter Haar, Introduction the Physics of Many-Body Systems (Interscience Publishers, Inc., 1958), Chap. 6.

2) G. K. Batchelor, The Theory of Homogeneous Turbulence (Cambridge University Press, 1953), p. 31.

3) E. J. Hopf, Rat. Mech. & Analysis 1 (1952), 87.

4) G. K. Batchelor, The Theory of Homogeneous Turbulence (Cambridge University Press, 1953), p. 32.

Chapter 2. A theory of the turbulent electric field fluctuations II.--- Energy spectral distribution.

§2.1 Foreword

In Chapter 1,[1] some characteristic properties of the turbulent electric field fluctuations with finite amplitude were studied making use of the method of redundant variables for many-body system. No consideration was taken of concerning the energy source responsible for the onset of the turbulent fluctuations; therefore, the structure of the spectral distribution has been left unsolved. In this chapter, a particular way of feeding energy onto the finite-amplitude is assumed and the spectral distribution (versus wave number) conforming to this condition is approximately determined.

The particular model we have in mind is, for example, a fully ionized plasma involved in a drift instability. This type of instability is caused by a unidirectional flow of electrons against ions. This has been studied extensively by various authors. But the studies carried out on this phenomenon are almost exclusively within the framework of linearized theory and we have very little information on the state which the plasma may attain after a finite lapse of time. It is said that the plasma will inevitably experience a turbulent state. The objective of the present study is to plough into the virgin soil of our knowledge and to discuss the matter qualitatively rather than quantitatively. We adopt a quasi-stationary model with a constant energy input at long wavelengths.

Longitudinal electric fields in an electron plasma are described by means of an assembly of harmonic oscillators corresponding to each wave number. In particular, the nonlinear effect can be regarded as the coupling between different oscillators. The description is equivalent to the conventional velocity distribution function method when the wave number takes on values ranging from zero to infinity. What is to be examined is the behavior of the longitudinal electric fields; the collective coordinates are suitable for this purpose. Therefore the most natural method for the present study is to examine the properties of the system of coupled oscillators whose variables are collective coordinates.

In a laboratory plasma, the energy due to density fluctuations is known to be a few per cent of the total energy. Therefore, average density fluctuations have an amplitude some twenty per cent of the basic density. Although a variety of models may be considered, a model of weakly-coupled harmonic oscillator system is permissible on first consideration, because of the fairly weak coupling.

The fact that the system to our interests can be expressed by a Hamiltonian equation suggests that the recent result by P. A. Sturrock[2)] --- action-transfer theory--- is useful. The theory gives the relations, in cases of small coupling energy, between eigen-frequencies, their shifts, their unperturbed energy and the coupling energy involving those degrees of freedom. As is usually the case, coupling energy is not very easily known. Therefore we reduce the problem in such a way that estimated frequency-shifts will lead us to know the energy partition among the degrees of freedom. In this way, the most cumbersome part of our work is the calculation of the frequency-shifts. The hypothesis of turbulence is introduced for longer wavelength region: namely, elementary waves constituting the observed fields are separate macroscopic waves in themselves, but after a long period of observation, they are of quasi-Gaussian property due to randomizing mode couplings.

By straight-forward procedure, the discussions can be extended and cover those important transport quantities, such as electrical and thermal conductivities, diffusion velocities, etc.

§2.2 Fundamental equations

When ions are assumed to form a uniform smeared-out background of positive charges, the Hamiltonian for electrons is

$$H = \sum p_i^2/2m + \frac{1}{2}\sum_{i\neq j} U(\vec{x}_i - \vec{x}_j), \qquad (2.1)$$

where $\vec{x}_i$ and $\vec{x}_j$ are the position vectors of the i^{th} and j^{th} electrons respectively, and U is the potential energy between these electrons. The summation covers all the possible values of integers i and j ($i\neq j$) corresponding to each electron (N electrons in all).

By a contact transformation defined by

$$\left.\begin{aligned} \rho_{\vec{k}} &= \frac{\partial G}{\partial \pi_{\vec{k}}}, \qquad \vec{p}_i = \frac{\partial G}{\partial \vec{x}_i}; \\ G &= \sum_{i,\vec{k}} \pi_{\vec{k}} \exp(-i\vec{k}\cdot\vec{x}_i), \end{aligned}\right\} \qquad (2.2)$$

we have

$$\rho_{\vec{k}} = \sum_i \exp(-i\vec{k}\cdot\vec{x}_i),$$

$$\vec{p}_i = -i\sum_{\vec{k}} \vec{k}\,\pi_{\vec{k}} \exp(-i\vec{k}\cdot\vec{x}_i).$$

The summation with respect to $\vec{k}$ is performed for all values of $\vec{k}$ including zero. Thus we have rigorously the following:

$$H = -\frac{1}{2m} \sum_{\vec{k},\vec{l}} (\vec{k}\cdot\vec{l})\, \pi_{\vec{k}}\, \pi_{\vec{l}}\, \rho_{\vec{k}+\vec{l}} + \sum_{\vec{k}} \frac{2\pi e^2}{k^2} \rho_{\vec{k}} \rho_{-\vec{k}} . \tag{2.3}$$

On regrouping the first term on the righ-hand side of the above equation into those for which $\vec{k} + \vec{l} = 0$ and those for which $\vec{k} + \vec{l} \neq 0$, we obtain the following equations:

$$\begin{aligned} H &= \frac{1}{2m} \sum_{\vec{k}} k^2 \pi_{\vec{k}} \pi_{-\vec{k}} + \sum_{\vec{k}} \frac{2\pi e^2}{k^2} \rho_{\vec{k}} \rho_{-\vec{k}} \\ &\quad - \frac{1}{2m} \sum_{\substack{\vec{k},\vec{l} \\ \vec{k}+\vec{l}\neq 0}} (\vec{k}\cdot\vec{l})\, \pi_{\vec{k}}\, \pi_{\vec{l}}\, \rho_{\vec{k}+\vec{l}} \\ &\equiv H^0 + H^I , \end{aligned} \tag{2.4}$$

$$H^0 = \frac{1}{2m} \sum_{\vec{k}} k^2 \pi_{\vec{k}} \pi_{-\vec{k}} + \sum_{\vec{k}} \frac{2\pi e^2}{k^2} \rho_{\vec{k}} \rho_{-\vec{k}} , \tag{2.5}$$

$$H^I = -\frac{1}{2m} \sum_{\substack{\vec{k},\vec{l} \\ \vec{k}+\vec{l}\neq 0}} (\vec{k}\cdot\vec{l})\, \pi_{\vec{k}}\, \pi_{\vec{l}}\, \rho_{\vec{k}+\vec{l}} . \tag{2.6}$$

H^0 stands for the set of independent harmonic oscillators, while

H^I for the interaction between them. For gaseous plasmas, some recent experimental evidence indicates that the mean amplitude of turbulent electric field fluctuations reaches 20 % of the base density, i.e., the interaction energy is roughly of the order of 10 % of the total oscillation energy.[3] On these grounds, we may take a model of weakly-coupled oscillators as still depicting the essentials of any strongly-agitated electron plasmas.

§2.3 Action transfer and frequency-shift

After P. A. Sturrock,[2] there are some distinguished properties for H^0 (assembly of harmonic oscillators) and H^I (interaction between them).

$$H = H^0 + H^I. \tag{2.7}$$

Now denoting suffixes by λ , etc. and normalizing, we are able to write

$$H^0 = \sum_{\lambda} \frac{1}{2}\left(p_\lambda^2 + \omega_\lambda^2 q_\lambda^2 \right) \tag{2.8}$$

where p_λ and q_λ are canonical variables and ω_λ the eigen-frequency. H^I is assumed to be expressed in a power series of p_λ and q_λ . On transforming variables by the relation

$$p_\lambda = (2J_\lambda \omega_\lambda)^{\frac{1}{2}} \cos\theta_\lambda , \quad q_\lambda = (2J_\lambda/\omega_\lambda)^{\frac{1}{2}} \sin\theta_\lambda \qquad (2.9)$$

into action and angle variables, we have

$$H^\circ = \sum_\lambda \omega_\lambda J_\lambda = \sum_\lambda E_\lambda , \quad (J_\lambda = E_\lambda/\omega_\lambda). \qquad (2.10)$$

E_λ is the energy belonging to a mode λ . Furthermore, we put

$$\theta_\lambda = \omega_\lambda t + \kappa_\lambda . \qquad (2.11)$$

Then, the interaction Hamiltonian is generally expressed in the following form:

$$H^I = \sum_{\substack{L\ell\lambda \\ Mm\mu \\ \vdots}} F_{L\ell\lambda Mm\mu\cdots} J_\lambda^{\frac{1}{2}L} J_\mu^{\frac{1}{2}M} \cdots \exp\{i\ell\{\omega_\lambda t + \kappa_\lambda\} + im\{\omega_\mu t + \kappa_\mu\} + \cdots\} \qquad (2.12$$

where L, M, ... are positive integers and

$$|\ell| \leq L , \qquad L + \ell = 0 \pmod 2) .$$

These relations are easily found by seeing that H^I is expressed in a power series of p_λ and q_λ . Thus the equation of motion in action-angle variables is

$$\left.\begin{aligned} \frac{dJ_\lambda}{dt} &= -\frac{\partial H^I}{\partial x_\lambda} , \\ \frac{dx_\lambda}{dt} &= \frac{\partial H^I}{\partial J_\lambda} . \end{aligned}\right\} \qquad (2.14)$$

Denoting the operation of taking mean values of some quantity by enclosing them with brackets like $\langle\ \rangle$, $\langle dx_\lambda/dt \rangle$ corresponding to the second equation of Eq.(2.14) gives mean frequency-shift. Regarding the contribution of (2.12) to d/dt of (2.14) as additive (as is indeed permissible for a weakly-coupled oscillators system), termwise analysis of d/dt may be denoted symbolically:

$$d/dt = \sum_{\substack{L\ell\lambda \\ Mm\mu \\ \cdots}} D_{L\ell\lambda Mm\mu\cdots} . \qquad (2.15)$$

Eq. (2.12), together with the second equation of Eq.(2.14), yields

$$\frac{\partial H^I}{\partial J_\lambda} = \frac{1}{2} \sum L F_{L\ell\lambda Mm\mu\cdots} J_\lambda^{\frac{1}{2}L-1} J_\mu^{\frac{1}{2}M} \cdots \exp\{i\ell(\omega_\lambda t + x_\lambda) + im(\omega_\mu t + x_\mu) + \cdots\}$$

$$\equiv \frac{dx_\lambda}{dt} = \sum D_{L\ell\lambda Mm\mu\cdots} x_\lambda .$$

If we write

$$H^{I}_{L\ell\lambda Mm\mu\cdots}$$

in place of

$$F_{L\ell\lambda Mm\mu\cdots} J_{\lambda}^{\frac{1}{2}L} J_{\mu}^{\frac{1}{2}M} \cdots \exp\{ i\ell(\omega_\lambda t + \varkappa_\lambda) + im(\omega_\mu t + \varkappa_\mu) + \cdots \},$$

then we have

$$J_\lambda \mathcal{D}_{L\ell\lambda Mm\mu\cdots} \varkappa_\lambda = \frac{1}{2} L H^{I}_{L\ell\lambda Mm\mu\cdots},$$

which can be written in a more general form as follows.

$$\frac{J_\lambda}{L} \mathcal{D}_{L\ell\lambda Mm\mu\cdots} \varkappa_\lambda = \frac{J_\mu}{M} \mathcal{D}_{L\ell\lambda Mm\mu\cdots} \varkappa_\mu = \cdots = \frac{1}{2} H^{I}_{L\ell\lambda Mm\mu\cdots}. \quad (2.16)$$

Now, we define $\Delta_1 \omega_\lambda$, $\Delta_1 \omega_\mu$, $\cdots$ to be the frequency shifts of the unperturbed eigen-frequency ω_λ , ω_μ , $\cdots$ due to an interaction $H^{I}_{L\ell\lambda Mm\mu\cdots}$. Thus we obtain

$$\frac{E_\lambda}{L} \cdot \frac{\Delta_1 \omega_\lambda}{\omega_\lambda} = \frac{E_\mu}{M} \cdot \frac{\Delta_1 \omega_\mu}{\omega_\mu} = \cdots = \frac{1}{2} H^{I}_{L\ell\lambda Mm\mu\cdots}. \quad (2.17)$$

The values of the positive intergers L, M, $\cdots$ depend on the way the variables p_λ , q_λ , p_μ , q_μ , ... are includ-ed in $H^I_{Ll\lambda Mm\mu\cdots}$. If the variables are not contained explic-itly in the interaction Hamiltonian, then, all those integers are zeros. Incidentally the total frequency-shift for mode λ is given by an equation like

$$\Delta\omega_\lambda = \frac{\omega_\lambda}{2E_\lambda}\sum L H^I_{Ll\lambda Mm\mu\cdots} . \qquad (2.18)$$

In Eq.(2.6), we write for simplicity

$$-\frac{1}{2m}(\vec{k}\cdot\vec{l})\,\pi_{\vec{k}}\,\pi_{\vec{l}}\,\rho_{\vec{k}+\vec{l}} = H^I_{\vec{k},\vec{l}} , \qquad (2.19)$$

by which Eq.(2.17) reduces to

$$E(\vec{k})\frac{\Delta_1\omega(\vec{k})}{\omega_p} = E(\vec{l})\frac{\Delta_1\omega(\vec{l})}{\omega_p} = \frac{1}{2}H^I_{\vec{k},\vec{l}} . \qquad (2.20)$$

We have noticed that every eigen-frequency for every harmonic oscillator in Eq. (2.5) is ω_p (angular frequency of electron plasma oscillations). $E(\vec{k})$ is the energy belonging to the wave number $\vec{k}$. $\Delta_1\omega$ denotes a partial shift due to an interaction $H^I_{\vec{k},\vec{l}}$.

By taking a time average on Eq. (2.20), we have

$$\mathcal{E}(\vec{k})\,\frac{\langle \Delta_1\omega(\vec{k})\rangle}{\omega_p} = \mathcal{E}(\vec{l})\,\frac{\langle \Delta_1\omega(\vec{l})\rangle}{\omega_p} = \frac{1}{2}\langle H^I_{\vec{k},\vec{l}}\rangle, \tag{2.21}$$

where $\mathcal{E}$ is for $\langle E\rangle$.

In the present study, the objective is to know the k-dependence of $\mathcal{E}(\vec{k})$. For this purpose, we have two choices from Eq.(2.21): either we estimate at $\langle \Delta_1\omega(\vec{k})\rangle$ and $\langle H^I_{\vec{k},\vec{l}}\rangle$ or we make a guess at $\langle \Delta_1\omega(\vec{k})\rangle$, $\mathcal{E}(\vec{l})$ and $\langle \Delta_1\,\omega(\vec{l})\rangle$. Usually the average value of the coupling energy is not easily known; therefore, we make the latter choice. In other words, we postulate that quantities at a wave number $\vec{l}$ are already known and calculate the frequency-shift at the relevant wave number $\vec{k}$, by the hypothesis of turbulent fluctuations, in order to determine the relative value $\mathcal{E}(\vec{k})$ approximately.

§2.4 Turbulent electric field fluctuations

What we call turbulent states are analogous to the turbulent phenomena treated in hydrodynamics. In the turbulent fluctuations (both in space and time), they can be analyzed into constituent elementary macroscopic waves (or in the hydrodynamic turbulence, int elementary eddies that correspond to each spatial Fourier component)

They can be thus made up from a set of waves at an arbitrary moment in time. But they are subject to a complex time evolution because of nonlinear mode couplings between numerous degrees of freedom, so that in the end they can be treated as statistical phenomena. It is indeed possible to follow their behavior within a very short time as an initial-value problem of dynamics. An average state, however, cannot be known by such a method.

Now we look on the phenomena as an aggregate of longitudinal waves with distinct wave numbers. Let the k-space (i.e. wave number space) be subdivided into the following four regions.

(I) $(0,k_0)$: The k_0 is a wave number at which the energy of the turbulent fluctuations is supplied or it gives the effective upper bound below which the energy is fed to the turbulent fluctuations. In other words, through this region (I), waves absorb energy from individual particles and are continually growing. Our model is that in which modes in the small wave number region are quasi-stationarily excited.

(II) (k_0,k_c): The definition of k_c is already given in the foregoing chapter, §1.3. In the long wavelength domain (k_0,k_c), we are able to neglect the particle aspect and mode-to-mode couplings determine the energy partition. The definition of k_c given in the preceding chapter is somewhat ambiguous; but, for the case of drift instability of the present concern, we have a large electron drift velocity, so that $\underline{v}$ appearing in the third equation of §1.3[4)] should be replaced by v_d (the drift velocity). v_d is about equal to or more than the thermal velocity of electrons. Moreover we should notice that when v_d is far larger than the thermal velocity, k_o becomes extremely small.

Thus we have (for a one-dimensional case)

$$k_c = \left(\frac{\varepsilon_t}{mnv_d^2} \cdot \frac{k_d}{k_o} \right)^{\frac{1}{2}} \cdot k_o$$

where

ε_t = the total overthermal fluctuation energy per unit volume.

$\frac{1}{2}mnv_d^2$ = the kinetic energy of the particles due to a uniform drift, per unit volume.

k_o = the main wave number at which the finite amplitude waves are strongly excited.

k_d = the Debye wave number.

(III) (k_c, k_d): We have in this region coexisting medium-like aspect and particle aspect. When this region has a long slope $(0 < k_c \ll k_d)$, then the energy density is rather low, so that the mode coupling effect may be neglected in the first approximation. This simplification is also self-consistent, since, on neglecting the mode coupling effect

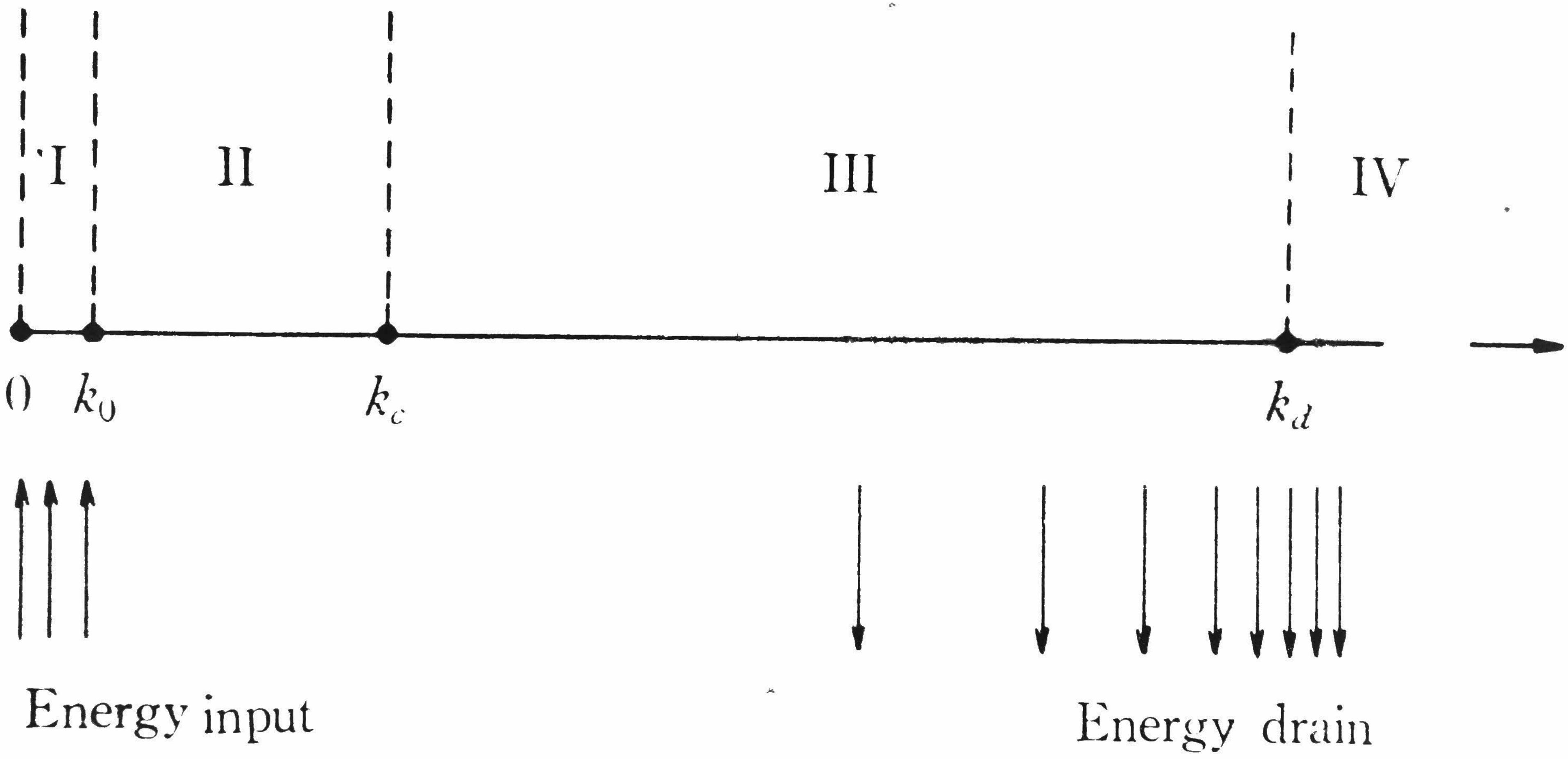

Fig. 2.1 Subdivision of the wave number space.

in this domain, the slope falls off as an inverse square of the wave number (as to be stated later) and thus rapidly approaches to zero.

(IV) (k_d, ∞): This region corresponds to scales shorter than the Debye wavelength. A Hamiltonian expression is possible including this region as expounded in §2.2. The fluctuations in these scales are such that we cannot define an eigen-frequency adequately. Namely, the coupling energy is of an order, almost the same as, or even more than, the unperturbed oscillator energy. We exclude this region from the scope of the present discussions.

We prescribe the state at the interfacing boundary of the regions (III) and (IV) as a given condition. And then we find the solution backward to the smallest-k region. Another premise that we should add is that the wave number domain involved is large enough to make the following condition valid:

$$0 < k_o \ll k_c \ll k_d.$$

§2.5 Spectral distribution in the region (III)

As has been stated in §2.4, when $k_c \ll k_d$ holds, the damping of fluctuations is largely overshadowed by the particle aspect, rather than the mode coupling due to finite amplitude effects. For this reason, the frequency corresponding to a wave number k ($k_c < k < k_d$) is given by the following equation. It is the dispersion relation with the Landau damping term. And even the dispersion relation discussed recently[5] with modifications to the unperturbed distribution gives the same result.

$$\omega^2(k) \simeq \omega_p^2 + 3\,\frac{\langle p^2 \rangle}{m^2}\,k^2. \tag{2.22}$$

$\langle p^2 \rangle$ is the mean square of the momentum of an electron. Let us notice

$$\langle p^2 \rangle = \frac{1}{N}\sum_{1}^{N} p_i^2 = -\frac{1}{N}\sum_{\vec{l}',\vec{l}''}(\vec{l}'\cdot\vec{l}'')\,\pi_{\vec{l}'}\,\pi_{\vec{l}''}\,\rho_{\vec{l}'+\vec{l}''}. \tag{2.23}$$

The frequency-shift arising from a mode coupling term,

$$H^{I}_{\vec{k},\vec{l}} \cong -\frac{1}{2m}(\vec{k}\cdot\vec{l})\,\pi_{\vec{k}}\,\pi_{\vec{l}}\,\rho_{\vec{k}+\vec{l}}, \tag{2.24}$$

is given by

$$\frac{E(\vec{k})}{\omega_p}\Delta_1\omega(\vec{k}) = \frac{E(\vec{l})}{\omega_p}\Delta_1\omega(\vec{l}) = -\frac{1}{4m}(\vec{k}\cdot\vec{l})\,\pi_{\vec{k}}\,\pi_{\vec{l}}\,\rho_{\vec{k}+\vec{l}} \tag{2.25}$$

Therefore, from Eqs.(2.22) and (2.23), the partial shift of frequency may be expressed as follows.*)

$$\left.\begin{aligned}\Delta_1 \omega(\vec{k}) &= -\frac{3}{2}\,\frac{k^2}{\omega_p m^2}\cdot\frac{1}{N}\,(\vec{k}\cdot\vec{l})\,\langle \pi_{\vec{k}}\,\pi_{\vec{l}}\,\rho_{\vec{k}+\vec{l}}\rangle,\\ \Delta_1 \omega(\vec{l}) &= -\frac{3}{2}\,\frac{l^2}{\omega_p m^2}\cdot\frac{1}{N}\,(\vec{k}\cdot\vec{l})\,\langle \pi_{\vec{k}}\,\pi_{\vec{l}}\,\rho_{\vec{k}+\vec{l}}\rangle.\end{aligned}\right\} \tag{2.26}$$

Thus, Eqs. (2.25) and (2.26) result in

$$k^2\,\mathcal{E}(\vec{k}) = l^2\,\mathcal{E}(\vec{l}). \tag{2.27}$$

---

*) The total shift of frequency $\Delta\omega(k)$ is $(3/2)\,k^2\langle p^2\rangle/(\omega_p$ from Eq.(2.22). $\langle p^2\rangle$ can be expressed by a sum of terms as is shown in Eq.(2.23); each of these terms now makes an additive contribution to the total shift. Therefore, if we refer to the partial shift due to one single coupling term defined by Eq.(2.24), then we are naturally led to Eq.(2.26) by comparing Eq.(2.24) with Eq.(2.23).

Since the vector $\vec{l}$ is not specified, we have, in the region (III):

$$\mathcal{E}(\vec{k}) \propto 1/k^2. \qquad (2.28)$$

From the above, the energy spectral distribution in the region (III) is determined, when $\mathcal{E}(\vec{k}_d)$ is given.

§2.6 Assumption of quasi-normality

In the wave number region (II), the mode coupling effect predominates. This is the region for which a macroscopic description is adequate. As the next step, we have to make the calculation of the frequency-shifts in this region. Prior to the calculation, we make an assumption that the variables π_k, ρ_k, etc. follow a quasi-normal statistical distribution and examine the properties of these variables.

Now we make averaging operations in the same way as expounded in the previous chapter.[1] Namely, the symbol $\langle\ \rangle$ represents the mean value for many samples taken by shifting the time origin arbitrarily. In this chapter, too, we have to see if higher order correlations can be decomposed to lower order correlations in order that the chain of the hierarchal system of two-time moment equations be cut and closed.

In order to define a characteristic function for the stochastic variables $\rho_{\vec{k}}$ and $\pi_{\vec{k}}$, let $y_{\vec{k}}$ and $z_{\vec{k}}$ be their corresponding arbitrary parameters respectively. Further we write, for simplicity, as follows:

$$[y, \rho] = \sum y_{\vec{k}} \rho_{\vec{k}}, \qquad [z, \pi] = \sum z_{\vec{k}} \pi_{\vec{k}}. \qquad (2.29)$$

We have

$$y_{\vec{k}}^{*} = y_{-\vec{k}}, \qquad z_{\vec{k}}^{*} = z_{-\vec{k}}, \qquad (2.30)$$

where " $*$ " stands for complex conjugate, and the characteristic function is

$$\Phi = E\left[\exp\left\{i\left([y, \rho] + [z, \pi]\right)\right\}\right] \qquad (2.31)$$

(E is for the expectation value).

The Hopf equation (see REF. 6) which this characteristic function should obey is

$$\frac{\partial \Phi}{\partial t} = \frac{i}{m} \sum_{\vec{k}} \sum_{\vec{n}} (\vec{k} \cdot \vec{n})\, y_{\vec{k}} \frac{\partial^2}{\partial y_{\vec{k}+\vec{n}}\, \partial z_{\vec{n}}} \Phi$$

$$- \frac{i}{2m} \sum_{\vec{k}} \sum_{\vec{h}} (\vec{k}-\vec{h}) \cdot \vec{h} \; z_{\vec{k}} \frac{\partial^2}{\partial z_{\vec{k}-\vec{h}} \, \partial z_{\vec{h}}} \, \Phi$$

$$- 4\pi e^2 \sum_{\vec{k}} \frac{1}{k^2} z_{\vec{k}} \frac{\partial}{\partial y_{-\vec{k}}} \Phi , \qquad (2.32)$$

where the sum includes the cases both $\vec{k}=0$ and $\vec{h}=0$.

In the following, we treat only those cases in which the statistical state is stationary in time; for instance, the time correlations are dependent on the time differences between the times at which the quantities are referred to, but independent of the time origin. Therefore,

$$\frac{\partial \Phi}{\partial t} = 0. \qquad (2.33)$$

By making use of the independence of different components, we may assume the following type of solutions to hold and study the sufficient conditions that the solutions are required to meet:

$$\frac{i}{m} \sum_{\vec{h}} (\vec{k} \cdot \vec{h}) \frac{\partial^2}{\partial y_{\vec{k}+\vec{h}} \, \partial z_{\vec{h}}} \Phi = A(\vec{k}) \, z_{\vec{k}} , \qquad (2.34)$$

$$- \left[\frac{i}{2m} \sum_{\vec{h}} (\vec{k}-\vec{h}) \cdot \vec{h} \frac{\partial^2}{\partial z_{\vec{k}-\vec{h}} \, \partial z_{\vec{h}}} + \frac{4\pi e^2}{k^2} \frac{\partial}{\partial y_{-\vec{k}}} \right] \Phi$$

$$= - A(\vec{k}) \, y_{\vec{k}} . \qquad (2.35)$$

If Eqs.(2.34) and (2.35) are satisfied by

$$\left.\begin{aligned}\Phi &= -c(\vec{k},\vec{h})\, y_{\vec{k}+\vec{h}}\, z_{\vec{h}}\, z_{\vec{k}}\,, \\ \Phi &= -d(\vec{k},\vec{h})\, y_{\vec{k}}\, z_{\vec{h}}\, z_{\vec{k}-\vec{h}} - e(\vec{k})\, y_{-\vec{k}}\, y_{\vec{k}}\end{aligned}\right\} \quad (2.36)$$

respectively (c, d, and e is independent of y and z), the only relation required is:

$$\frac{i}{2m}\sum_{\vec{k}}(\vec{k}-\vec{h})\cdot\vec{h}\, d(\vec{k},\vec{h}) + \frac{4\pi e^2}{k^2}\, e(\vec{k}) = \frac{i}{m}\sum_{\vec{h}}(\vec{k}\cdot\vec{h})\, c(\vec{k}, \quad (2.37)$$

We then find that

$$\Phi = 1 - \sum e(\vec{k})\, y_{\vec{k}}\, y_{-\vec{k}} - \sum g(\vec{k},\vec{h})\, y_{\vec{k}+\vec{h}}\, z_{\vec{h}}\, z_{\vec{k}} \quad (2.38)$$

$$\left(g(\vec{k},\vec{h}) = c(\vec{k}-\vec{h},\vec{h}) + d(\vec{k},\vec{h}) \right)$$

is a solution for (2.32) and (2.33).

Now in general, Φ is expressible as a functional power series in y and z. In particular, on noticing that y and z may take arbitrary complex values, it can be ascertained that those terms, such as yz, yyy and yyz cannot be contained in this functional power series for the present case. This means that concerning the joint distribution of ρ and π we have the following relations:

$$\left.\begin{array}{l} \langle \pi\rho\rangle \sim 0, \\ \langle \rho\rho\rho\rangle \sim 0, \\ \langle \pi\rho\rho\rangle \sim 0. \end{array}\right\} \qquad (2.39)$$

On summing up the above results, we know the first few terms of the functional power series for Φ . Therefore, for y_k and z_k not very large in absolute magnitude, we have

$$\Phi \sim \exp\Big\{ -\sum e(\vec{k})\, y_{\vec{k}}\, y^{*}_{\vec{k}} - \sum g(\vec{k},-\vec{k})\, z_{\vec{k}}\, z^{*}_{\vec{k}} - \sum_{\vec{k}+\vec{h}\neq 0} g(\vec{k},\vec{h})\, y_{\vec{k}+\vec{h}}\, z_{\vec{k}}\, z_{\vec{h}} \Big\}. \qquad (2.40)$$

In the above, $e(\vec{k})$ and $g(\vec{k},-\vec{k})$ $(0 < |\vec{k}| < \alpha)$ give kinetic and potential energy of the harmonic oscillators respectively, while $g(\vec{k},\vec{h})$ is the energy of coupling.

§2.7 Energy spectral distribution in the region (II)

In this section calculations are made of the frequency-shift for an arbitrary wave number $\vec{k}$ in the region (k_o, k_c) making use of the hypothesis of turbulent fluctuations. Now let us take another wave number $\vec{l}$ that is also in this region. (The vector $\vec{k}+\vec{l}$ is a non-zero vector which lies in the same wave number region.) The calculations are performed along the way indicated by Eq.(2.20) or Eq.(2.21). In other words, particular attention is paid to the coupling term $\langle H^I_{\vec{k},\vec{l}} \rangle$ to which vectors $\vec{k}$ and $\vec{l}$ make a joint contribution, and the partial shift of eigen-frequency, $\Delta_1 \omega$, due to this single coupling is calculated. The actual total shift of frequency, $\Delta \omega$, is understood to be the sum of all the partial shifts of frequency due to those terms of H^I involving the specific wave number $\vec{k}$.

$$\mathcal{E}(\vec{k}) \frac{\langle \Delta_1 \omega(\vec{k}) \rangle}{\omega_p} = \mathcal{E}(\vec{l}) \frac{\langle \Delta_1 \omega(\vec{l}) \rangle}{\omega_p} = \frac{1}{2} \langle H^I_{\vec{k},\vec{l}} \rangle , \tag{2.41}$$

$$\begin{aligned} \mathcal{E}(\vec{k}) \frac{\sum \langle \Delta_1 \omega(\vec{k}) \rangle}{\omega_p} &= \mathcal{E}(\vec{l}) \frac{\sum \langle \Delta_1 \omega(\vec{l}) \rangle}{\omega_p} \\ &= \mathcal{E}(\vec{k}) \frac{\langle \Delta \omega(\vec{k}) \rangle}{\omega_p} = \sum_{\vec{k}', \vec{l}'} \frac{1}{2} \langle H^I_{\vec{k}', \vec{l}'} \rangle , \end{aligned} \tag{2.42}$$

where the summation in the right of the last equation covers those terms in which either $\vec{k}'$, $\vec{l}'$ or $\vec{k}'+\vec{l}'$ coincides with $\vec{k}$.

In a way similar to the one we have followed in the preceding chapter, by employing the hypothesis of quasi-normal distribution, we have the following equation for the total frequency-shift (See Appendix 2 for its derivation):

$$\frac{d}{dt}\langle \rho_{\vec{k}}\, \rho^{*0}_{\vec{k}} \rangle = -\frac{1}{m}\sum_{\vec{h}} (\vec{k}\cdot\vec{h})\langle \pi_{\vec{h}}\, \rho_{\vec{k}+\vec{h}}\, \rho^{*0}_{\vec{k}} \rangle + \frac{N}{m} k^2 \langle \pi_{-\vec{k}}\, \rho^{*0}_{\vec{k}} \rangle \tag{2.43}$$

$$\frac{d}{dt}\begin{pmatrix} \langle \pi_{\vec{h}}\, \rho_{\vec{k}+\vec{h}}\, \rho^{*0}_{\vec{k}} \rangle \\ \langle \pi_{-\vec{h}-\vec{k}}\, \rho_{-\vec{h}}\, \rho^{*0}_{\vec{k}} \rangle \\ \langle \rho_{-\vec{h}}\, \rho_{\vec{k}+\vec{h}}\, \rho^{*0}_{\vec{k}} \rangle \\ \langle \pi_{-\vec{k}-\vec{h}}\, \pi_{\vec{h}}\, \rho^{*0}_{\vec{k}} \rangle \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\frac{4\pi e^2}{h^2} & \frac{N}{m}(\vec{k}\cdot\vec{h}) \\ 0 & 0 & -\frac{4\pi e^2}{(\vec{h}+\vec{k})^2} & \frac{N}{m} h^2 \\ \frac{N}{m} h^2 & \frac{N}{m}(\vec{k}+\vec{h})^2 & 0 & 0 \\ -\frac{4\pi e^2}{(\vec{k}+\vec{h})^2} & -\frac{4\pi e^2}{h^2} & 0 & 0 \end{pmatrix}$$

$$\times \begin{pmatrix} \langle \pi_{\vec{h}}\, \rho_{\vec{k}+\vec{h}}\, \rho^{*0}_{\vec{k}} \rangle \\ \langle \pi_{-\vec{h}-\vec{k}}\, \rho_{-\vec{h}}\, \rho^{*0}_{\vec{k}} \rangle \\ \langle \rho_{-\vec{h}}\, \rho_{\vec{k}+\vec{h}}\, \rho^{*0}_{\vec{k}} \rangle \\ \langle \pi_{-\vec{k}-\vec{h}}\, \pi_{+\vec{h}}\, \rho^{*0}_{\vec{k}} \rangle \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{m}\left[(\vec{h}\cdot\vec{k})\langle \rho_{\vec{k}+\vec{h}}\, \rho^{*}_{\vec{k}+\vec{h}} \rangle - (\vec{k}+\vec{h})\cdot\vec{k} \times \langle \rho_{\vec{h}}\, \rho^{*}_{\vec{h}} \rangle\right] \\ -\frac{1}{2m}\left[\sum_{\vec{l}} l^2 \langle \pi_{-\vec{l}}\, \pi_{\vec{l}} \rangle - 2(\vec{h}\cdot\vec{k})\langle \pi_{-\vec{h}}\, \pi_{\vec{h}} \rangle + 2(\vec{h}+\vec{k})\cdot\vec{k}\langle \pi_{\vec{h}+\vec{k}}\, \pi_{-\vec{h}-\vec{k}} \rangle\right] \end{pmatrix}$$

$$
\times \langle \pi_{-\vec{k}} \rho^{*0}_{\vec{k}} \rangle + \begin{pmatrix} \frac{1}{m} (\vec{k}+\vec{h})\cdot\vec{h} \langle \pi_{-\vec{h}} \pi_{\vec{h}} \rangle \\ \frac{1}{m} (\vec{k}+\vec{h})\cdot\vec{h} \langle \pi_{-\vec{h}-\vec{k}} \pi_{\vec{h}+\vec{k}} \rangle \\ 0 \\ 0 \end{pmatrix} \langle \rho_{\vec{k}} \rho^{*0}_{\vec{k}} \rangle, \tag{2.44}
$$

$$
\frac{d}{dt} \langle \pi_{-\vec{k}} \rho^{*0}_{\vec{k}} \rangle = -\frac{1}{2m} \sum_{\vec{r}} (\vec{k}+\vec{r})\cdot\vec{r} \langle \pi_{-\vec{k}-\vec{r}} \pi_{\vec{r}} \rho^{*0}_{\vec{k}} \rangle - \frac{4\pi e^2}{k^2} \langle \rho_{\vec{k}} \rho^{*0}_{\vec{k}} \rangle \tag{2.45}
$$

From the above equation, we select out the contribution due to

$$
\delta H^{I}_{\vec{k},\vec{l}} = -\frac{1}{2m} (\vec{k}\cdot\vec{l})\, \pi_{\vec{k}} \pi_{\vec{l}} \rho_{\vec{k}+\vec{l}} - \frac{1}{2m} (\vec{k}-\vec{l})\cdot\vec{l}\, \pi_{\vec{k}-\vec{l}} \pi_{\vec{l}} \rho_{\vec{k}} . \tag{2.46}
$$

For that purpose, we only have to pick the terms $\vec{h}=\vec{l}$, $\vec{l}=\vec{l}$ and $\vec{r}=\vec{l}$ from the sums $\sum_{\vec{h}}$, $\sum_{\vec{l}}$ and $\sum_{\vec{r}}$ respectively; here the last two correspond to the term on the right of Eq.(2.46) By utilizing Eq.(2.39) as initial conditions and carrying out a Laplace transformation on Eqs.(2.43) through (2.45), we seek the frequency-shifts, isolating those terms corresponding to Eq.(2.46)

Taking $\vec{k}$ for a fixed vector, we examine the intensity of interactions $H^{I}_{\vec{k},\vec{l}}$ for different $\vec{l}$. We have

$$
| H^{I}_{\vec{k},\vec{l}} | \sim | \vec{k}\cdot\vec{l}\; \pi_{\vec{k}} \pi_{\vec{l}} \rho_{\vec{k}+\vec{l}} |
$$

$$\sim |\vec{k}+\vec{l}| \{ \mathcal{E}(\vec{k}) \cdot \mathcal{E}(\vec{l}) \cdot \mathcal{E}(\vec{k}+\vec{l}) \}^{1/2},$$

where an approximate relation

$$k^2 \langle \pi_{\vec{k}} \pi^*_{\vec{k}} \rangle \sim \frac{1}{k^2} \langle \rho_{\vec{k}} \rho^*_{\vec{k}} \rangle \sim \mathcal{E}(\vec{k})$$

is made use of. Therefore we can conclude that interactions with larger $\vec{l}$ are stronger ($|\vec{l}| \gg |\vec{k}|$). This is indeed compatible with the conclusion which will be drawn from the statement; namely, by considering only those $\vec{l}$'s for which $|\vec{l}| \gg |\vec{k}|$, it will be shown later that $\mathcal{E}(\vec{k})$ in the region (II) is proportional to $|\vec{k}|$, and therefore

$$|H^{I}_{\vec{k},\vec{l}}| \sim (k+l)^{3/2} (kl)^{1/2} \sim k^{\frac{1}{2}} l^2.$$

Thus the assumption that we should be mainly concerned about the $\vec{l}$ ($|\vec{l}| \gg |\vec{k}|$) is not in contradiction to the results.

Therefore, let $\vec{l}$ be such that $|\vec{l}| \gg |\vec{k}|$; then, by the approximate calculations shown in Appendix 3. we derive the following equation for the one-dimensional case, where the shift due to δH^{I} is denoted by $\Delta_2 \omega(\vec{k})$ for $\vec{k}$, and $\Delta_2 \omega(\vec{l})$ for $\vec{l}$. Thus, we are now able to show the energy spectral distribution <u>versus</u> the wave number k.

$$\Delta_2\,\omega(\vec{k}) \cong \frac{1}{12\omega_p m}\cdot\frac{\ell^3}{k}\,\mathcal{E}(\ell)\ ; \tag{2.47}$$

$$\mathcal{E}(k)\,\frac{\Delta_2\omega(k)}{\omega_p} = \mathcal{E}(\ell)\,\frac{\Delta_2\omega(\ell)}{\omega_p} \equiv K(\ell)\,, \tag{2.47'}$$

$$\therefore\quad \mathcal{E}(k) = 12\,m\,\omega_p^2\,\frac{K(\ell)}{\ell^3\,\mathcal{E}(\ell)}\cdot k \quad (>0) \tag{2.48}$$

When $\vec{k}\cdot\vec{\ell}>0$, then $K(\ell)<0$, <u>vice versa</u>; see Eq.(A 15). Energy distribution for long wavelengths in the wave number region (II) is known by adding up the contributions due to various $\vec{l}$ ($|\vec{l}| \gg |\vec{k}|$). In this way, we can find out $\mathcal{E}$(k) from Eq.(2.48) by specifying the states at the shorter wavelength $\vec{l}$. Namely, for the present case, $\mathcal{E}$(k) is proportional to $|\vec{k}|$.

See Appendix 4 for the discussion of Eq. (2.47').

§2.8 Energy partition in the regions (II) and (III)

On recalling the definition of k_c , we are able to draw the Fig. 2.2 .

The general features seen in the figure are rather understandable. In the region where the mode-coupling predominates, strong oscillations excited at the wave number k_o , or in its neighbourhood are more easily coupled to the larger wave numbers.

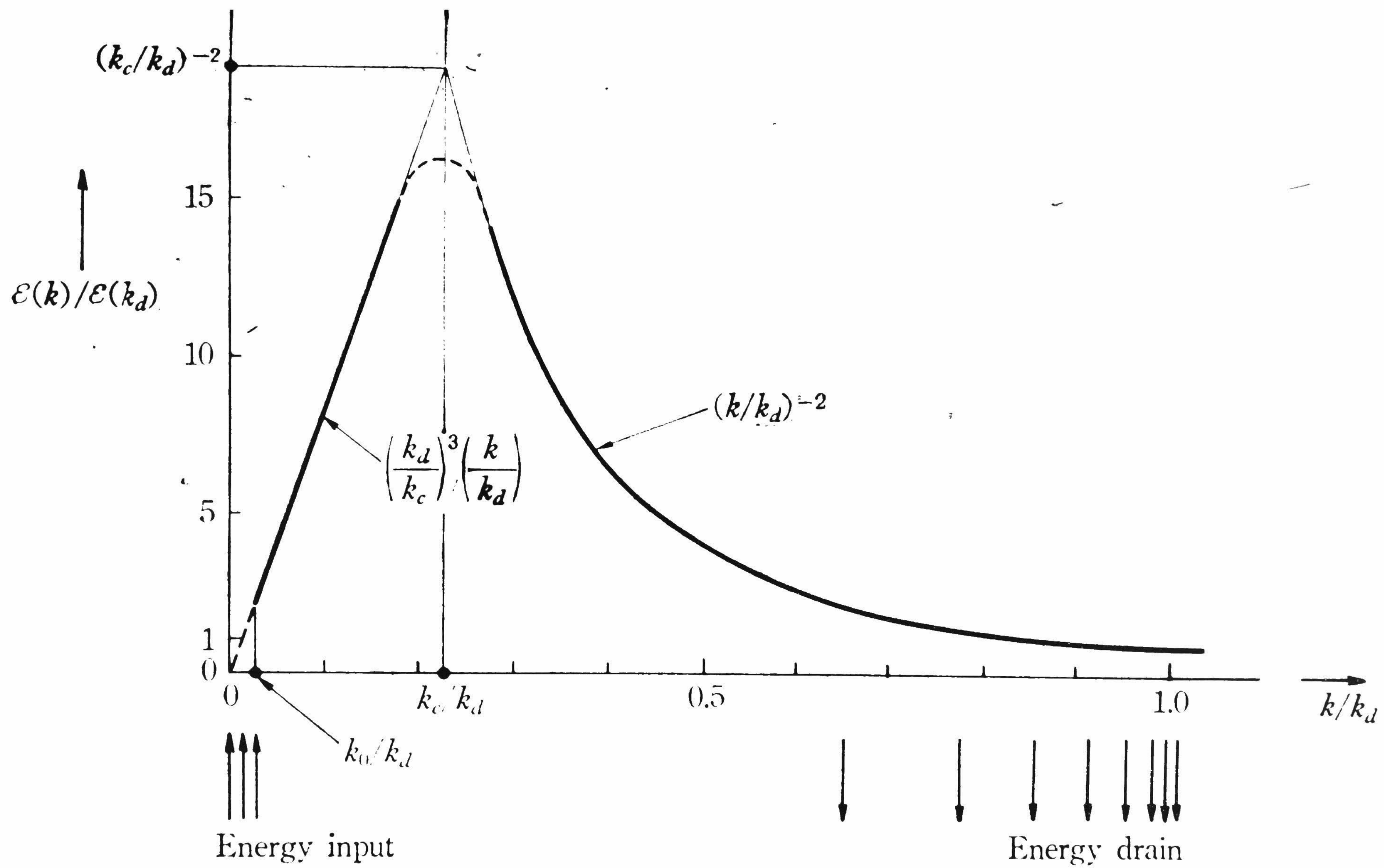

Fig. 2.2. Spectral energy distribution of the one-dimensional turbulent electric fi fluctuations versus wave number.

Therefore, the lifetime of fluctuations on increasing k becomes shorter, but the energy is more effectively transferred to the larger k. This is the reason why we have a rising characteristic. As the particle aspects become influential on increasing k, the energy of the turbulent fluctuations is transferred to the individual particles and this process comes to be very outstanding in the neighborhood of the Debye wave number. Finally we may conclude that the spectral distribution changes its character in the neighborhood of the critical wave number k_c.

§2.9 Concluding remarks

We have thus attempted to determine the energy spectral distribution of the turbulent electric field fluctuations in electron plasmas <u>versus</u> wave numbers, by utilizing a theory of a weakly-coupled harmonic oscillator system. The results obtained will be important clues to the study of the effective transport phenomena in overthermal turbulent states. Moreover, an electron plunging into the region of overthermal electric field fluctuations in hot plasmas is very likely to be affected more by the turbulent fluctuations than by the ordinary collision events. This suggests that the turbulent electric field fluctuations are in some cases fatal to the confinement of thermonuclear hot plasmas. To develop a theory of confinement including all these effects will need another lengthy chapter.

References (Chapter 2)

1) T. Tsuda, Prog. Theor. Phys. 28 (1962), 489.

2) P. A. Sturrock, Ann. of Phys 9 (1960), 422.

3) S. Yoshikawa and D. J. Rose, Phys. Fluids 5 (1962), 334.

4) T. Tsuda, Prog. Theor. Phys. 28 (1962), 489. See particularly pp. 494 and 495.

5) W. E. Drummond and D. Pines, "Non-Linear Stability of Plasma Oscillations," Conference on Plasma Physics and Controlled Nuclear Fusion Research, Salzburg (Sept., 1961), CN-10/134. See esp. Eq. (A.9) in the Appendix.

6) E. J. Hopf, J. Ratl. Mech. & Anal. 1 (1952), 87.

PART 2

DIFFUSION OF PLASMAS ACROSS THE MAGNETIC FIELD

Chapter 3. Diffusion of plasmas across the magnetic field

§3.1 Introduction

Suppose there were a strong and uniform magnetic field. Charged particles experience collisions less frequently, as the temperature is raised degree by degree: they finally become "frozen" to the line of magnetic force and the plasma ceases its motion in bulk across the magnetic field. In this state, the density gradient, perpendicular to the line of force, is sustained steadily. It is indeed true that, in this idealized situation, the two-body Coulomb force acting as "impacts" between the particles comes to be much less influential. On the other hand, it is open to question if we can neglect the effect of irregular electric fields due to other charged particles within the Debye sphere. The charged particles at the center of the sphere would find that these electric fields are of duration, much longer than the aforesaid two-body force, although the irregular fields are weaker than the impact forces. It is probable that these electric fields, with the coexisting magnetic field, cause the charges drift randomly across the magnetic field thus resulting in the bulk motion of the plasma. Particularly, the (turbulent) electric field fluctuations with a period longer than the period of gyration may be responsible for the "drain diffusion", which should be distinguished from the classical diffusion process.

These phenomena should be treated by some statistical theory with many-body correlations properly considered. But in this chapter we assess the effects just mentioned, by the assumption of the independence of each individual particle, for the limiting case of very hot plasmas. Namely, we follow the Markovian method.[2)]

§3.2 Electric field fluctuation and the motion of guiding centers

The plasma is supposed to be very dilute. Let the strong magnetic field $\vec{B}$ directed uniformly along the z-axis. Some gradient in density is assumed to exist in the x-direction; guiding centers would drift and diffuse in this x-direction. Furthermore, the temperature is assumed to be spatially uniform. We discuss the spatial diffusion.

Let the position of the guiding of a test particle be $\vec{R}$ (X, Y, Z) with reference to the rest coordinate system. Take the vectors $\vec{r}_0$ and $\vec{v}$ for its particle position and velocity respec-tively. Then, we have

$$\vec{R} = \vec{r}_0 + \frac{m_t}{q_t B^2} \vec{v} \times \vec{B} , \qquad (3.1)$$

where m_t and q_t ($\gtrless 0$) are the mass and the charge of the test particle. The equation of motion for the particle in the directionn vertical to the magnetic field is given by

$$\frac{d\vec{v}_\perp}{dt} = \frac{q_t}{m_t} (\vec{E}_\perp + \vec{v}_\perp \times \vec{B}) . \qquad (3.2)$$

$\vec{v}_\perp$ and $\vec{E}_\perp$ are the velocity and the electric-field components perpendicular to the magnetic field. Therefore,

$$\Delta \vec{R}_\perp = \frac{d\vec{R}_\perp}{dt} \Delta t$$

$$= \frac{1}{B^2} \left\{ (\vec{E}_\perp \Delta t) \times \vec{B} \right\}. \qquad (3.3)$$

In the above, $\vec{E}_\perp$ is again the component of the fluctuating electric field due to field particles vertical to the line of magnetic force. Δt is the mean duration of the fluctuation for the observer moving with the test charge. $\Delta \vec{R}_\perp$ is the distance traversed by the guiding center in the interval of time Δt. The relation of Eq.(3.3) holds irrespective of

a (gyration radius) $\gtrless$ ρ_D (the Debye radius)

or a $\gtrless$ $n^{-1/3}$ ($\equiv$ d).

n is the number density of the charges.

The flux of guiding centers diffusing in the x-direction is given by the following equation:

$$\bar{F} = n(x)\langle \Delta x \rangle - \frac{1}{2}\frac{\partial}{\partial x}\left(n(x)\langle (\Delta x)^2 \rangle \right), \qquad (3.4)$$

where the brackets $\langle \quad \rangle$ stand for the mean value per unit time.

When we think of binary collision only, in Eq.(3.3), and the change of velocity is supposed to occur impulsively due to $q_t \vec{E}_\perp$ in an infinitesimal interval of time Δt (much shorter than the gyration period), then we have the same results as were obtained impact-theoretically by Rosenbluth and Longmire.[1]

§3.3 Electric field fluctuations within the Debye sphere

Let us consider the distance r_s by which any pair of particles are said to have approached closely enough to form a binary system. In our hot plasma, r_s is much shorter than the interparticle distance d; therefore the electric field acting on the testt particle is Holtsmark-like, i.e. the electric field has a rather long period of fluctuation, although it is weak.[2] We have

$$kT_e \simeq kT_i = eEr_s = e\frac{ec^2}{r_s^2}r_s \qquad (3.5)$$

where T_e and T_i are the electron and the ion temperatures respectively, k is the Boltzmann constant. For example, for $n = 10^{13}\ \mathrm{cm}^{-3}$ and $T_e = T_i = 10^6\ {}^\circ\mathrm{K}$,

$$r_s \sim 10^{-9}, \quad d = n^{-\frac{1}{3}} \sim 5 \times 10^{-3}$$

$$\therefore \quad r_s \ll d. \qquad (3.6)$$

We introduce the following assumptions concerning the stochastic properties of the electric field fluctuations. The fields are due to those charged particles within the Debye sphere.

i) The probability that a charged particle occupies an infinitesimal volume $d\vec{r}$ in the Debye sphere with radius ρ_D at a certain moment is given by $d\vec{r} / (\frac{4}{3}\pi \rho_D^3)$.

ii) The motions of the field particles are independent of each other.

iii) Numerous field particles, say, more than 1,000, exist in the Debye sphere.

Using the assumptions i) and ii), we can calculate, by Markoff's method, the intensity of the electric field fluctuations at the point under consideration (i.e. at the position of the test particle). We know the classical theory by Holtsmark[2),3)] in which the electric field fluctuations due to charged particles uniformly distributed in space are calculated for specific point in space. In this theory, the probability density is calculated of the electric field fluctuation by the Markovian method: first, with the radius of the sphere being finite, then in the process of simplification the radius is taken to be infinitely large. Therefore, it is expected that the results thus obtained may coincide with the results which are obtained with a large enough radius to include numerous field particles. Indeed, G. Ecker[4)] made a numerical analysis with the finite radius throughout the calculation, and found that the results are similar to those obtained by Holtsmark, as long as the field particles are numerous (more than 10^3) within the Debye sphere.

Therefore, if the assumption iii) holds, then the field fluctuations are Holtsmark-like. On these grounds, the Holtsmark theory may be applied to the hot plasma with density gradients.

Since the probability distribution of the instantaneous value of the electric field fluctuation is uniform within the Debye sphere (see i) and ii)), Eq.(3.3) yields

$$\langle \Delta X \rangle = \frac{1}{B^2} (\vec{E}_1 \times \vec{b}) = 0, \qquad (3.7)$$

where the overbar stands for a mere averaging. Therefore, we have to see to

$$\langle (\Delta X)^2 \rangle = \frac{1}{B^2} \overline{E_y^2 \Delta t} . \qquad (3.8)$$

As the next step, the probability density $W(\vec{E}, \vec{f})$ for $\vec{E}$ to fall into the interval $(\vec{E}, \vec{E}+d\vec{E})$ and for $\vec{f}$ $(= d\vec{E}/dt$: the rate of field fluctuation) to fall into $(\vec{f}, \vec{f}+d\vec{f})$ must be calculated. This function, $W(\vec{E}, \vec{f})$ is known by the Markovian method in principle. But it is sufficient for the present purpose to assume

$$\overline{E_y^2 \Delta t} \sim \overline{E_y^2 \, \overline{\Delta t}_{\vec{E}, \vec{f}}} . \qquad (3.9)$$

$\overline{\Delta t_{\vec{E},\vec{v}}}$ is the expectation value of the lifetime for a pair of fixed values $\vec{E}$ (fluctuating field) and $\vec{v}$ (the velocity of the test particle). We are able to make use of the values $\overline{\Delta t_{\vec{E},\vec{v}}}$ obtained by S. Chandrasekhar and J. von Neumann[2) 5)] within the range appropriate for the present problem.

We define the following quantities:

$$\left.\begin{aligned} Q_H &\equiv \frac{e c^2}{r_0^2} = e c^2 \left(\frac{4\pi n}{3}\right)^{2/3}, \\ r_0 &= \left(\frac{3}{4\pi n}\right)^{1/3}; \\ \beta &\equiv |\vec{E}| / Q_H . \end{aligned}\right\} \qquad (3,10)$$

The probability density of the event that $|\vec{E}|$ falls ...... the interval $(|\vec{E}|, |\vec{E}| + d|\vec{E}|)$ is given by

$$\left.\begin{aligned} W(|\vec{E}|) &= H(\beta)/ Q_H , \\ H(\beta) &= \frac{2}{\pi\beta} \int_0^\infty x \sin x \, e^{-\left(\frac{x}{\beta}\right)^{\frac{3}{2}}} dx . \end{aligned}\right\} \qquad (3,11)$$

The mean duration of the fluctuating electric field observed by the test particle in motion, i.e. the lifetime of the fluctuation, is calculated as follows. We have to distinguish the following two cases, in relation to the magnitudes of the radius of gyration a_e and the mean interparticle distance d. Namely, we have two cases:

CASE a) $a_e > d$ or $a_e \approx d$;

CASE b) $a_e \ll d$.

In the case of b), it becomes probable that, by the random cross-field drifting motion, the electrons are able to "drain" through other particles. In the case of a), the velocity of the test particle has some relation to the lifetime just mentioned, when observed by the test particle itself.

The diffusion due to mutual interactions among like particle is much less important, so that this can be neglected as a higher-order process. In what follows, the results are given for various cases.

CASE a) $a_e > d$ or $a_e \approx d$ (test particles = electrons).

By means of the general formula,[2) 5)]

$$\overline{\Delta t_{\vec{E},\vec{v}}} = \frac{d}{\sqrt{\overline{|\vec{u}|^2}}}\left(\frac{\beta^{\frac{3}{2}} H(\beta)}{G(\beta)}\right)^{\frac{1}{2}}\left[1 + \frac{|\vec{v}|^2}{\overline{|\vec{u}|^2}} + \frac{5}{12\pi}\frac{|\vec{v}|^2}{\overline{|\vec{u}|^2}}\frac{H(\beta)}{\beta^{\frac{1}{2}}G(\beta)}\right]^{-\frac{1}{2}} \tag{3.12}$$

($\vec{v}$ = the velocity of the test particle; $\vec{u}$ = the velocity of the field particles), we have

$$\overline{\Delta t}_{\vec{E},\vec{v}_e} = \frac{|\vec{E}|}{\sqrt{\overline{|\vec{f}|^2}}_{\vec{E},\vec{v}_e}} = \frac{d}{\sqrt{v_e^2}} \left[\left(\frac{4}{15}\right)^{\frac{1}{3}} (2\pi) \frac{\beta^{\frac{3}{2}} H(\beta)}{G(\beta)} \cdot \frac{1}{1 + \frac{5}{12\pi} \cdot \frac{H(\beta)}{\beta^{\frac{1}{2}} G(\beta)}} \right]^{\frac{1}{2}}$$

(3.12′)

where $\vec{v}_e$ is the velocity of the test electron. Generally the lifetime is related to both the velocity of the test particles and the velocity of the field particles. In the above, it is assumed that

$$\sqrt{\overline{|\vec{u}_i|^2}} \; / \; v_e \longrightarrow 0 \qquad (3.13)$$

($\vec{u}_i$ = the velocity of the ions).

$G(\beta)$ is defined by

$$G(\beta) = \frac{3}{2} \int_0^{\beta} \beta^{-\frac{3}{2}} H(\beta)\, d\beta . \qquad (3.14)$$

CASE a) (test particles = ions)

Since we have approximately,

$$v_i' \Big/ \sqrt{\overline{|\vec{u}_e|^2}} \longrightarrow 0 \qquad (3.15)$$

the lifetime is as follows:

$$\overline{\Delta t}_{\vec{E},\vec{v}_i} = \frac{|\vec{E}|}{\sqrt{|\vec{f}|^2}} = \frac{d}{\sqrt{|\vec{u}_e|^2}}\left[\left(\frac{4}{5}\right)^{\frac{1}{3}}(2\pi)\,\frac{\beta^{\frac{3}{2}}H(\beta)}{G(\beta)}\right]^{\frac{1}{2}}. \qquad (3.16)$$

In Eqs.(3.12') and (3.16), the isotropy of the velocity distribution is assumed. If we regard the value of the term

$$1\Big/\left(1+\frac{5}{12\pi}\,\frac{H(\beta)}{\beta^{\frac{1}{2}}G(\beta)}\right)^{1/2} \qquad (3.17)$$

as unity*), then we have from Eqs.(3.12') and (3.16)

$$\langle(\Delta x)^2\rangle \cong \frac{1}{3}\,\frac{a_H^2}{B^2}\int_0^{\beta_s}\beta^2 H(\beta)\,\overline{\Delta t}\,d\beta, \qquad (3.18)$$

where

$$\beta_s = (d/r_s)^2. \qquad (3.19)$$

---

*) N. B.

$(3.17) \to 1 \quad (\beta\to 0 \text{ or } \beta\to\infty),$

$(3.17) \simeq 1 \quad (\beta \simeq 1).$

The upper bound for the integral in Eq.(3.18) has been taken to be β_s . This value cannot be determined no more accurately than in the corresponding case of the ordinary correlation method. Thus we have

$$\langle (\Delta X)^2 \rangle \simeq \frac{1}{3} \left[\left(\frac{4}{15}\right)^{\frac{1}{3}} (2\pi) \right]^{\frac{1}{2}} \frac{a_H^2}{B^2} \frac{d}{\sqrt{|\vec{u}_e|^2}} \times \int_0^{\beta_s} \beta^2 H(\beta) \left[\frac{\beta^{\frac{3}{2}} H(\beta)}{G(\beta)} \right]^{\frac{1}{2}} d\beta . \qquad (3.20)$$

For a particular case where

$$\int_0^{\beta_s} \beta^2 H(\beta) \left[\frac{\beta^{\frac{3}{2}} H(\beta)}{G(\beta)} \right]^{\frac{1}{2}} d\beta = O(\log \beta_s) \qquad (3.21)$$

$$(\beta_s \gg 1)$$

holds, so that, neglecting the factor of an order of unity. we obtain the following:

$$\langle (\Delta X)^2 \rangle \sim \frac{a_H^2}{B^2} \frac{d}{u_e} \log \beta_s . \qquad (3.22)$$

CASE b) (test particles = electrons)

For the sake of simplicity, we assume that the ions obey an isotropic velocity distribution. For the value of $\vec{v}$ in Eq.(3.12), the value such that

$$|\vec{v}| = a_H \beta / B \tag{3.23}$$

should be taken. Denote the value of β by β_c, which satisfies

$$\frac{(a_H \beta / B)^2}{\overline{|\vec{u}_i|^2}} \simeq 1 .$$

Then obviously we have

$$\beta_c^2 = \frac{B^2}{a_H^2} \cdot \overline{|\vec{u}_i|^2} . \tag{3.24}$$

Again neglect the factor of order of unity, Eq.(3.12) yields

$$\left.\begin{aligned} \overline{\Delta t} &\sim \frac{B}{a_H} d \cdot \beta^{-\frac{3}{2}} \quad \cdots\cdots\cdots \quad \beta_c \ll \beta \\ \overline{\Delta t} &\sim \frac{d}{\sqrt{\overline{|\vec{u}_i|^2}}} \cdot \beta \quad \cdots\cdots \quad \beta \lesssim \beta_c \end{aligned}\right\} \tag{3.25}$$

Since $\beta \sim 1.6$ gives the maximum occurrence of the fluctuating electric field, we have the following results:

$$\begin{cases} \langle (\Delta X)^2 \rangle \sim \dfrac{(1.6\,Q_H)^2}{B^2} \cdot \dfrac{B}{Q_H}\, d \cdot 1.6^{-\frac{3}{2}} = 1.6^{\frac{1}{2}} \dfrac{Q_H\, d}{B} & (\beta_c \ll 1.6) \\ \langle (\Delta X)^2 \rangle \sim \dfrac{(1.6\,Q_H)^2}{B^2} \cdot \dfrac{d}{\sqrt{\overline{|\vec{u}_i|^2}}} \cdot 1.6 = 1.6^3 \dfrac{Q_H^2\, d}{B^2 \sqrt{\overline{|\vec{u}_i|^2}}} & (\beta_c \gtrsim 1.6) \end{cases} \tag{3.26}$$

CASE b) (test particles = ions)

The electrons, now acting as the field particles, are assumed to follow the next relation

$$T_{e_{\parallel}} \ll T_{e_{\perp}}$$

with respect to their kinetic temperature, in the planes parallel and vertical to the line of magnetic force. The lifetime is determined by their motion in the plane vertical to the line of magnetic force.

$$\overline{\Delta t} = \begin{cases} \left(\dfrac{\beta^{\frac{3}{2}} H(\beta)}{G(\beta)} \right)^{\frac{1}{2}} \dfrac{d}{|\vec{v}_i|} \dfrac{1}{\sqrt{1 + \dfrac{H(\beta)}{\beta^{\frac{1}{2}} G(\beta)}}}\,, & \left(\dfrac{Q_H}{B} \ll |\vec{v}_i| \right); \\ \left(\dfrac{\beta^{\frac{3}{2}} H(\beta)}{G(\beta)} \right)^{\frac{1}{2}} \cdot \dfrac{d}{Q_H / B}\,, & \left(\dfrac{Q_H}{B} \gg |\vec{v}_i| \right). \end{cases} \tag{3.27}$$

Therefore we have

$$\langle(\Delta X)^2\rangle = \begin{cases} \dfrac{a_H^2}{B^2}\cdot\dfrac{d}{|\vec{v}_i|}\displaystyle\int_0^{\beta_s}\beta^2 H(\beta)\left(\frac{\beta^{\frac{3}{2}}H(\beta)}{G(\beta)}\right)^{\frac{1}{2}}\frac{d\beta}{\sqrt{1+\dfrac{H(\beta)}{\beta^{\frac{1}{2}}G(\beta)}}}, \\ \qquad\qquad\qquad\qquad (|\vec{v}_i| \gg a_H/B) \\ \dfrac{a_H}{B}\,d\displaystyle\int_0^{\beta_s}\beta^2 H(\beta)\left(\frac{\beta^{\frac{3}{2}}H(\beta)}{G(\beta)}\right)^{\frac{1}{2}}d\beta, \quad (|\vec{v}_i| \ll \frac{a_H}{B}), \end{cases} \tag{3.28}$$

The above relations are further simplified as follows, if $\beta_s \gg 1$:

$$\langle(\Delta X)^2\rangle \sim \begin{cases} \dfrac{a_H^2}{B^2}\dfrac{d}{|\vec{v}_i|}\log\beta_s, & |\vec{v}_i| \gg \dfrac{a_H}{B}; \\ \dfrac{a_H}{B}\,d\log\beta_s, & |\vec{v}_i| \ll \dfrac{a_H}{B}. \end{cases} \tag{3.29}$$

§3.4 Resume

By means of the formulae so far obtained, the flux of the diffusing particles can be calculated from the equation

$$F = -\frac{1}{2}\langle(\Delta X)^2\rangle\frac{\partial N}{\partial X}.$$

It is anticipated that in some of the cases above mentioned the controlling magnetic field becomes less effective in order to sustain a given density gradient. This occurs in the limiting case of a low-density plasma with a high confining field.

REFERENCES (Chapter 3)

1) C. L. Longmire & M. N. Rosenbluth, Phys. Rev. 103, 507 (1956).

2) S. Chadrasekhar, Rev. Mod. Phys., 15,1 (1943), esp. Chap. IV.

3) A. Unsöld, Physik der Sternatmosphären, Springer-Verlag (1955), 308--311.

4) G. Ecker, Z. f. Physik, 148, 593 (1957).

5) S. Chandrasekhar & J. von Neumann, Ap. J., 97, 1 (1943).

PART 3 (Chapter 4)

APPLICATION OF THE MONTE CARLO METHOD TO SYSTEMS OF NON-LINEAR ALGEBRAIC EQUATIONS

---- PARTICLE PHYSICS ANALOGUE INTRODUCED INTO THE DOMAIN OF NUMERICAL ANALYSIS ----

§4.1 Introduction

It is, in general, a difficult problem to find all the roots of a given system of nonlinear algebraic equations with several unknowns, even though we meet with such demands in many fields of applied sciences. We have several methods for such problems, which have been extended from some powerful method for algebraic equations with a single unknown. Iterative procedures, thus developed, are often found unsuccessful, since they do not necessarily converge, given arbitrary starting values. The conventional methods of iteration are guaranteed against divergence, only if the starting value is well enough in the neighborhood of the solution.[1]

Thus, as long as we have no means to place all the roots even very roughly, we have no working method to find all the roots of a given system of nonlinear algebraic equations. In the present paper, a new scheme of the Monte Carlo technique is proposed in order to find approximate locations of these roots, which are believed to provide successful starting values for the current classical methods of iteration. The results of a test on a digital computer are also illustrated, together with a sketch of the actual computer procedures.

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[1]See Todd [1], pp. 255 -- 278.

Apart from the novelty of the underlying concept, the method has a merit that it can treat cases with many variables with no particular increase in complexity.

§4.2 Principle

Suppose that a system of nonlinear algebraic equations

$$F_i(x_1, x_2, \cdots, x_n) = 0, \quad (i = 1, 2, \cdots, n) \qquad (4.1)$$

is given. In principle, the F_i's may be any functions of the variables x_1, x_2, ..., x_n. However, in the present context, the F_i's are regarded as polynomials of the n variables. Complex roots may be considered, as indeed suggested in the last section: but, for simplicity, only the real roots are taken into account.

Corresponding to Eq. (4.1), the following pseudo-potential U is introduced:

$$U(x_1, x_2, \cdots, x_n) = \sum_{i=1}^{n} a_i^2 F_i^2(x_1, x_2, \cdots, x_n),$$

$$(a_i = \text{constants}). \qquad (4.2)$$

Let an aggregate of particles move about in this hypothetical n-dimensional potential field, colliding with each other. The manner of collision is arbitrary; at least a certain mean free path must be specified. There is no mutual interaction among particles during free flights; the particles are in motion conforming to the potential, given by Eq. (4.2), between collisions. When a collision occurs, the transfer of both the momentum and the energy are supposed to result in.

When a statistical equilibrium is established, it is what the statistucal mechanics tells us[2] that the number density distribution of the particles in the 2n-dimensional phase space is proportional to an exponential factor

$$exp\ (-E/\beta),$$

where β is a constant equivalent to the kinetic temperature. E is the total energy, namely the sum of the potential energy U and the kinetic energy T of the particle under consideration. Thus

$$E = U(x_1, x_2, \cdots, x_n) + T(\dot{x}_1, \dot{x}_2, \cdots, \dot{x}_n). \tag{4.3}$$

(Here dots represent differentiations with respect to the time.) In particular the distribution in the $(x_1, x_2, \ldots, x_n)$-coordinate space is proportional to the weight factor

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[2]See Landau [2], Chap. 3.

$$\exp(-U/\beta),$$

thus the probability that a particle is found in the interval between $(x_1, x_2, \ldots, x_n)$ and $(x_1+dx_1, x_2+dx_2, \ldots, x_n+dx_n)$ is given by

$$P(x_1, x_2, \cdots, x_n)\, dx_1\, dx_2 \cdots dx_n$$

$$= \frac{e^{-U(x_1, x_2, \cdots, x_n)/\beta}\, dx_1\, dx_2 \cdots dx_n}{\int\int\cdots\int e^{-U(x_1, x_2, \cdots)/\beta}\, dx_1\, dx_2 \cdots dx_n}. \tag{4.4}$$

Therefore, by help of the ergodicity, after having observed the behavior of a particle for a time sufficiently long to bring about a statistical equilibrium, then the particle would have occupied most frequently the part of the space where

$$\nabla U(x_1, x_2, \cdots, x_n) = 0 \tag{4.5}$$

holds. In terms of particle ensemble, the particles would have shown the most dense concentration at the position where Eq. (4.5) holds. The points where we have Eq. (4.5) are the real roots of the given system of equations, as seen from Eq. (4.2).

The equations of motion for the particle are

$$\left.\begin{aligned} dp_{x_i}/dt &= -\partial U/\partial x_i = -U_{x_i}(x_1, \cdots, x_n), \\ p_{x_i} &= dx_i/dt, \qquad (i = 1, 2, \cdots, n) \end{aligned}\right\} \tag{4.6}$$

We cumulatively observe the quantities

$$\langle x_i \rangle = \frac{\int \cdots \int x_i e^{-U/\beta} dx_1 dx_2 \cdots dx_n}{\int \cdots \int e^{-U/\beta} dx_1 dx_2 \cdots dx_n} \tag{4.7}$$

having the particle in motion. If the steepness of the gradient of the Boltzmann factor is remarkable, then the relation

$$U(\langle x_1 \rangle, \langle x_2 \rangle, \cdots, \langle x_n \rangle) \cong 0 \tag{4.8}$$

is found valid, so that these observed quantities $(\langle x_1 \rangle, \langle x_2 \rangle, \cdots, \langle x_n \rangle)$ afford us the knowledge of approximate locations of the roots now looked for. One method to have the steepness of the gradient outstanding is to prescribe that the scale factors a_i's in Eq. (4.2) be properly large. It will indeed be demonstrated in a later section that these simple tricks work notably for the purpose of steepening the gradient and smoothing out the wrinkles where the particle may be trapped unfavorably for some time.

In actual procedures, the differentials in Eq. (4.6) are replaced with differences; the particle moves obeying the difference equations thus constructed. Let the mean free path be λ and make decision whether a collision has occurred or not at each step. Here a step is defined as $s\lambda$ of the traversed distance. s is a positive number sufficiently smaller than unity. In the event of collisions, the transfer of kinetic energy takes place and the particle acquire a new level of energy qT_m , which is an arbitrary fraction of the upper bound of the kinetic energy T_m. The particle involved in collision in this way is scattered isotropically. The q just mentioned is a pseudo-random number equidistributed in the interval [0, 1]. The collision probability at each step of $s\lambda$ is shown in Fig. 4.1. The model is such that when the kinetic energy surpasses a threshold value T_m, then the particle makes a transition to lower energy-level and is scattered. Therefore, it is expected that by thus restricting the range of movement of the representative point in the phase space, at least along the axes of momenta, the state of the statistical equilibrium could achieved in a relatively small number of steps.

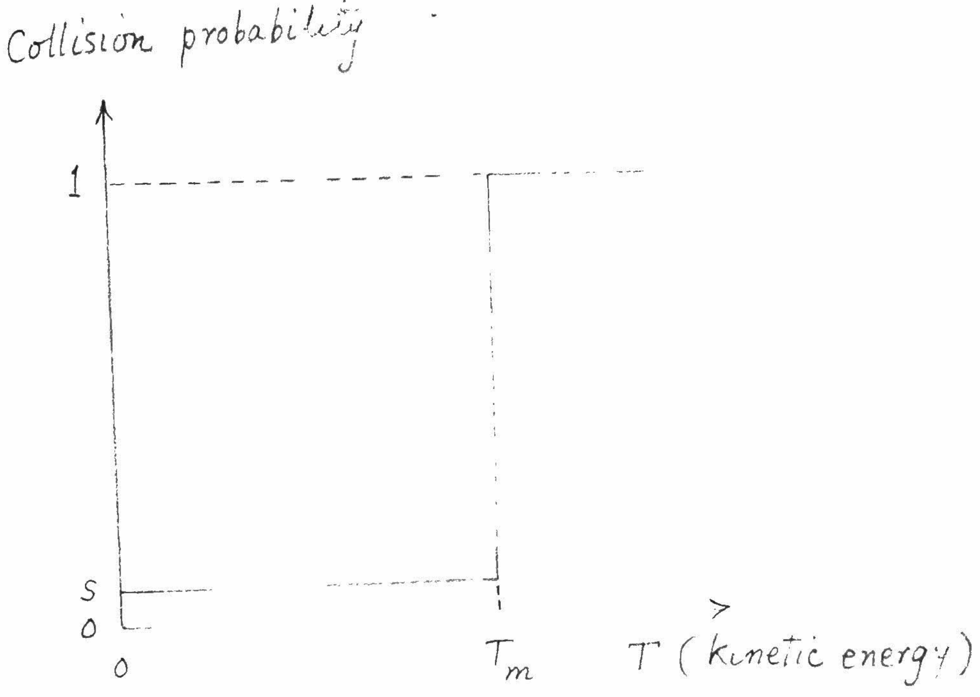


Fig. 4.1

At the preliminary stage of computation, $\langle x_i \rangle$ and $\langle (x_i - \langle x_i \rangle)^2 \rangle$ are observed with the passage of time. If the latter turns out to be non-decreasing, then the roots are distributed wide apart, $\langle x_i \rangle$ giving the center of distance of those roots. In the case where we have several distinct roots, the anisotropy of the spatial distribution now mentioned can be inferred from the inequality among the values $\langle (x_i - \langle x_i \rangle)^2 \rangle$ (i=1, 2, ..., n). Namely this quantity $\langle (x_i - \langle x_i \rangle)^2 \rangle$ is the largest, say, for i= ν , along the axis along which the roots are most diversely located. As the next step the observation of $\langle x_i \rangle$ is done distinguishing $\langle x_i \rangle$ (i=1, ... , n), such that $x_\nu < \langle x_\nu \rangle$, from $\langle x_i \rangle$ (i=1, ... , n), such that $x_\nu > \langle x_\nu \rangle$. Subdividing the entire space in this way, there exist more than one root in each of these subdivided regions. Continuing making such binary subdivisions in this way, an ultimate state will be reached where in each subdivision the variances $\langle (x_i - \langle x_i \rangle)^2 \rangle$ diminish steadily and the limiting values of $\langle x_i \rangle$ will give the approximate values of the roots.

In the present scheme of computation, the range of sea^r^ching need not be restricted at the outset. However, in the example shown later, the range of searching is specified for the sake of convenience, i. e. only the pseudo-random numbers equidistributed in the unit interval [0, 1] were utilized. In many of the practical cases, we can usually specify the upper and lower bounds loosely between which the roots are supposed to exist.

Moreover it should be noted as for the range of searching that by the inversion of x_i into $1/x_i$ all the points outside the unit hypercube are replaced with those inside the hypercube.

§4.3 Algorithm

The example which will be shown in the following section is of the form

$$\left.\begin{aligned} F(x, y) &= 0, \\ G(x, y) &= 0. \end{aligned}\right\} \qquad (4.9)$$

Therefore, the computational procedures suitable for the solution of the above equation will be stated. But similar procedures are followed in cases of many more vareables.

(0°) Define T_m, s, d, N_1, N_2 ($\gg N_1$).

$0 \rightarrow N$, $0 \rightarrow x$, $0 \rightarrow x^2$, etc.

$1 \rightarrow a_i^2$ ($i=1, 2, \dots, n$).

(1°) Choose the initial position randomly in the present subdivision, generating uniformly distributed random numbers.

(2°) Determine the value of the kinetic energy:

$$qT_m \rightarrow T,$$

where q is symbolically written for random numbers generated whenever necessary, which are equi-distributed in the interval [0,1]. This notation of q is preferably used in what follows.

(3°) Find the velocity:

$$\sqrt{T}\cos 2\pi q \rightarrow p_x , \quad \sqrt{T}\sin 2\pi q \rightarrow p_y .$$

(4°) Compute the time required of the particle for one step's flight:

$$s\lambda / \sqrt{T} \longrightarrow dt .$$

(5°) Compute the displacement:

$$p_x \cdot dt \longrightarrow dx , \quad p_y \cdot dt \longrightarrow dy .$$

(6°) Compute the increments of the velocity:

$$- U_x \cdot dt \longrightarrow dp_x , \quad - U_y \, dt \longrightarrow dp_y .$$

(7°) Find the new position and velocity after the displacement:

$$x + dx \rightarrow x , \quad y + dy \rightarrow y ;$$

$$p_x + dp_x \rightarrow p_x , \quad p_y + dp_y \rightarrow p_y .$$

(8°) If the representative point (x, y) is not in the relevant subdivision, then (2°), else (9°).

(9°) Record:

$$x + \sum x \rightarrow x , \quad y + \sum y \rightarrow y .$$

$$x^2 + \sum x^2 \to x^2, \quad y^2 + \sum y^2 \to y^2;$$

$$N + 1 \longrightarrow N.$$

(10°) If $N=0 \pmod{N_1}$, then (20°), else (11°).

(11°) If $p_x^2 + p_y^2 > T_m$, then (2°), else (12°).

(12°) If $q > s$, then (4°) (i. e. no collision), else (2°) (i. e. collision).

(20°) Compute, store, and write:

$$\sum x/N \to \langle x \rangle \quad (\equiv MX),$$

$$\sum y/N \to \langle y \rangle \quad (\equiv MY);$$

$$\left.\begin{aligned} \sum x^2/N - \langle x^2 \rangle &\to DEVX, \\ \sum y^2/N - \langle y^2 \rangle &\to DEVY. \end{aligned}\right\} \text{(variance)}$$

(21°) If $n=0 \pmod{N_2}$, then (30°), else (11°).

(30°) Clear the counter of N:

$$0 \to N.$$

(31°) If DEVX $> d^2$ or DEVY $> d^2$ and either of them non-decreasing, then (40°), else (32°).

(32°) Take up the next subdivision if any, and go to (1°), else halt.

(40°) If DEVX $>$ DEVY, then (41°), else (42°).

(41°) Divide the present subdivision further into halves, one in which $x > MX$, the other in which $x < MX$. Go to (43°).

(42°) Divide the present subdivision further into halves, one in which $y > MY$, the other in which $y < MY$. Go to (43°).

(43°) Steepen the gradient of the potential:

$$r_a a_i^2 \longrightarrow a_i;$$

lessen the volume of the momentum space:

$$r_t T_m \longrightarrow T_m;$$

make the step finer so as to gain more resolving power:

$$r_\lambda \lambda \longrightarrow \lambda.$$

Here r_a, r_t and r_λ are predetermined constants; $r_a > 1$, $0 < r_t < 1$, $0 < r_\lambda < 1$, otherwise rather arbitrary.

§4.4 Example

As an example, the equations

$$\left.\begin{aligned} F(x, y) &\equiv xy - 3/16 = 0, \\ G(x, y) &\equiv x^2 + y^2 - 5/8 = 0 \end{aligned}\right\} \qquad (4.10)$$

are considered. The potential is denoted as follows:

$$U(x, y) = a^2(F^2 + G^2). \qquad (4.11)$$

The roots of Eq. (4.10) are:

$$(x, y) = (0.25, 0.75) \text{ and } (0.75, 0.25). \tag{4.12}$$

The range of searching is now understood to be $0 \leq x \leq 1$, $0 \leq y \leq 1$. See Fig. 4.2. In this example, we have potential minimum, which does not correspond to the roots, in the middle of the square domain.

To find approximate values of the roots of this sample problem, the computation was started with constants defined as such:

$$T_m = 4.0,$$

$$s = 0.1,$$

$$\lambda = 0.1,$$

$$d = 0.1.$$

Other factors were taken to be

$$r_a = 10,$$

$$r_t = 1/400,$$

$$r_\lambda = 1/10.$$

The results of computation of the first searching are shown in Table 4.1 and Fig. 4.3. Checking was done at the procedure (31^o) and the interval of x, [0, 1], was halved into [0, 0.462···(=MX)] and [0.462···, 1]. The Monte Carlo search was performed in these subdivisions separately. Further results thus obtained are shown

FIGURE CAPTIONS

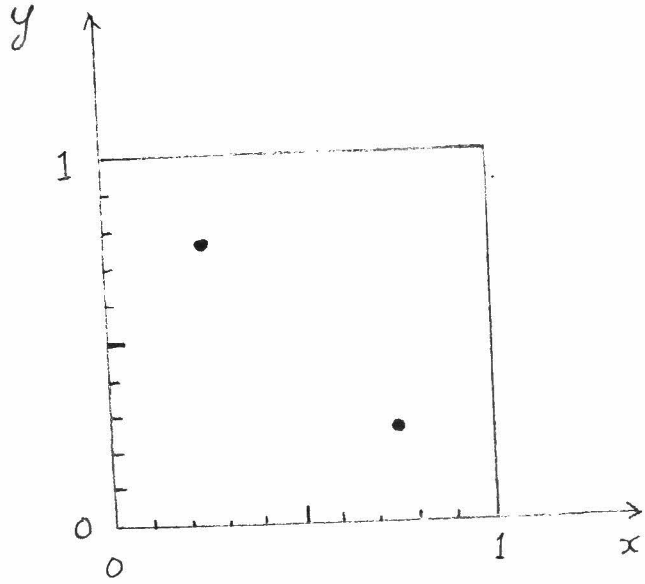


Fig. 4.2

Table 4.1 Preliminary survey in $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$

N	MX	MY	DEVX	DEVY
200	0.71987	0.21580	0.2888:-1	0.7988:-2
400	0.44739	0.16199	0.9722:-1	0.9994:-2
600	0.33461	0.19379	0.9297:-1	0.1128:-1
800	0.41663	0.19029	0.1026:0	0.9630:-2
1000	0.49035	0.19294	0.1060:0	0.9164:-2
1200	0.53655	0.24704	0.1015:0	0.2763:-1
1400	0.56922	0.33252	0.9580:-1	0.6839:-1
1600	0.55110	0.40750	0.9857:-1	0.9973:-1
1800	0.50223	0.45179	0.1074:0	0.1052:0
2000	0.46242	0.45675	0.1112:0	0.9818:-1

N. B. 0.2888:-1 = 0.2888×10^{-1}, etc.

$T_m = 4.0$, $s = 0.1$, $\lambda = 0.1$, $d = 0.1$, $a^2 = 1.0$.

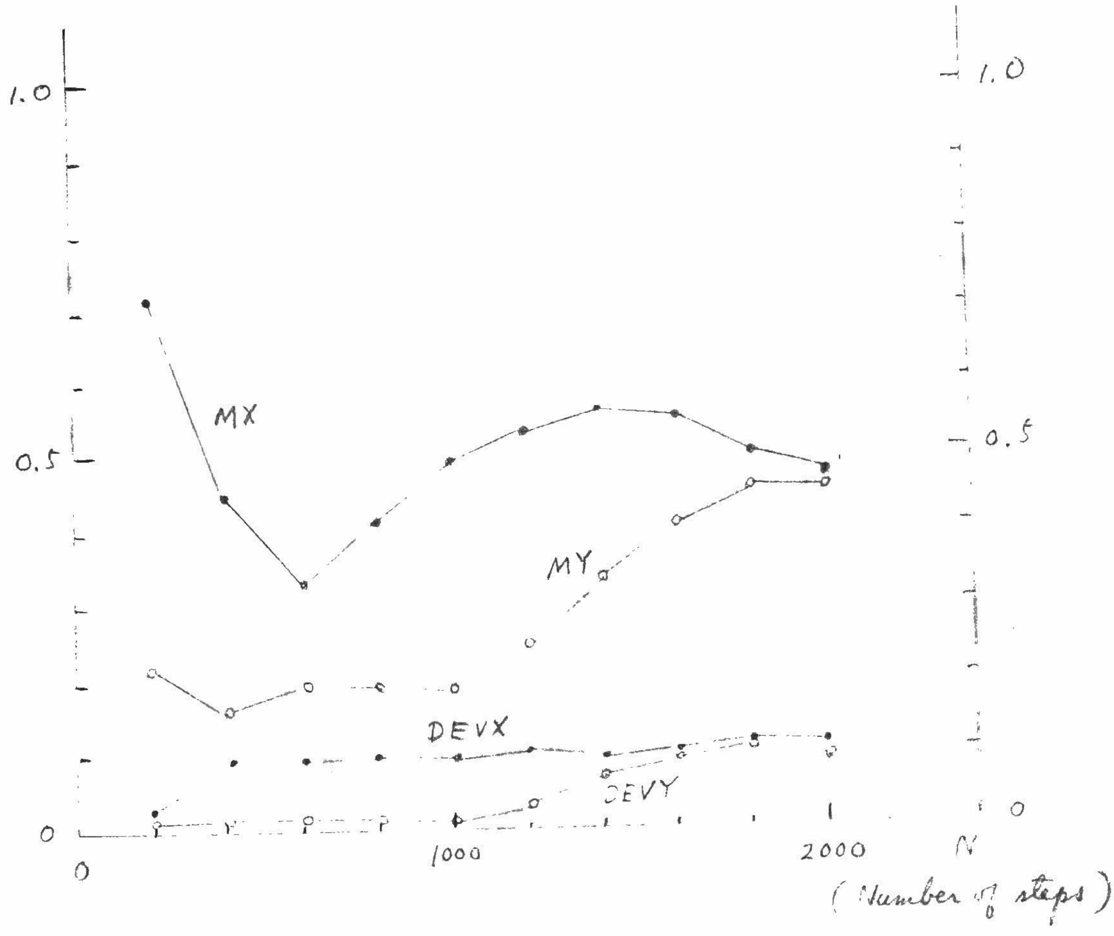


Fig 4.3

Table 4.2 Survey in the subdivision $0 \leq x \leqq 0.462...,$

$0 \leqq y \leqq 1$

N	MX	MY	DEVX	DEVY
1000	0.27933	0.74968	0.2302:-2	0.3295:-3
2000	0.26768	0.74872	0.1641:-2	0.2374:-3
3000	0.26485	0.74799	0.1187:-2	0.1824:-3
4000	0.26051	0.74806	0.1021:-2	0.1706:-3
5000	0.26020	0.74721	0.8408:-3	0.1552:-3

N. B. $T_m = 0.01$, $s = 0.1$, $\lambda = 0.01$, $a^2 = 10$.

Table 4.3 Survey in the subdivision $0.462... \leq x \leqq 1,$

$0 \leqq y \leqq 1$

N	MX	MY	DEVX	DEVY
1000	0.74360	0.15622	0.4704:-3	0.2616:-2
2000	0.74856	0.19945	0.3152:-3	0.3260:-2
3000	0.75193	0.21002	0.2672:-3	0.2467:-2
4000	0.75004	0.22312	0.2402:-3	0.2469:-2
5000	0.75053	0.22722	0.2073:-3	0.2083:-2
5800	0.75047	0.23029	0.1865:-3	0.1890:-2

N. B. $T_m = 0.01$, $s = 0.1$, $\lambda = 0.01$, $a^2 = 10$.

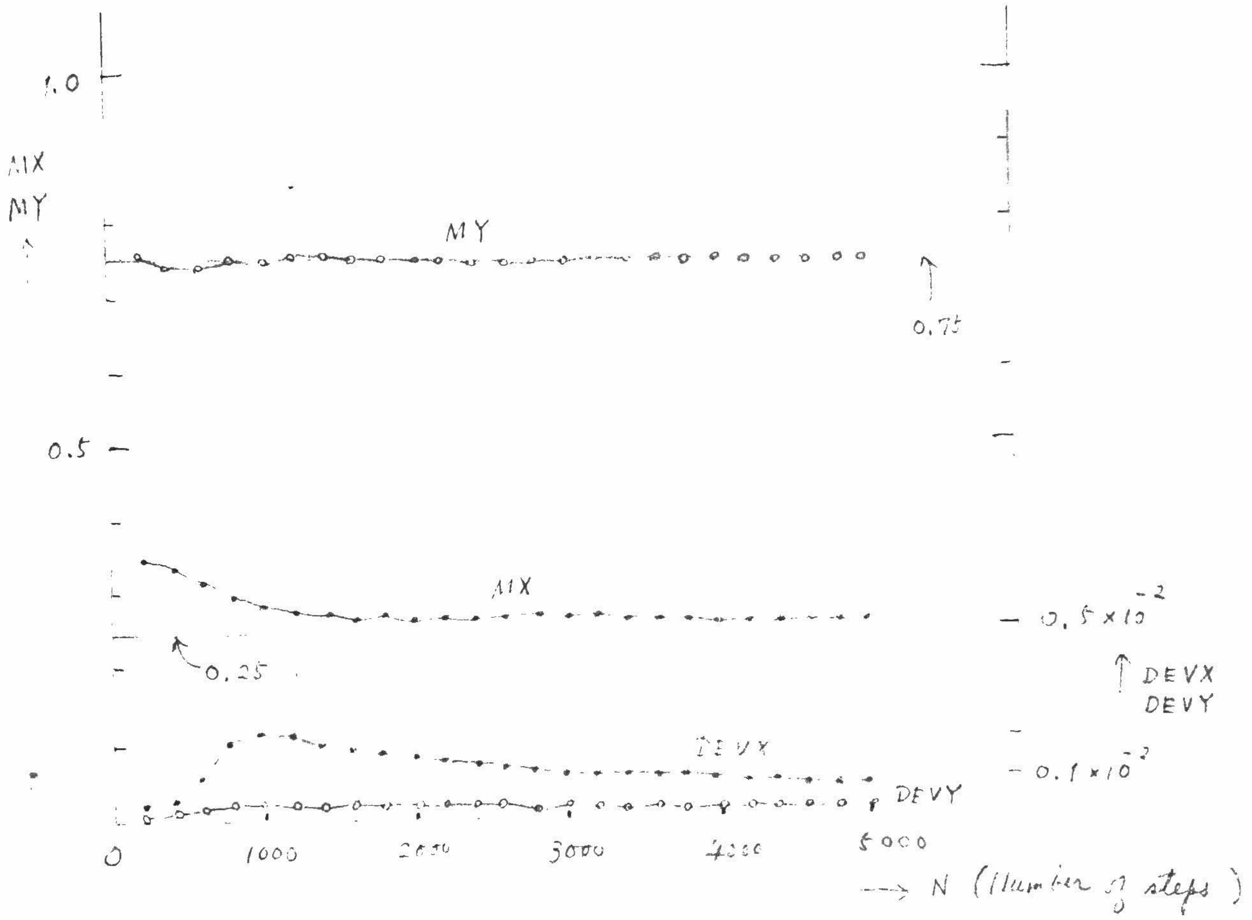

1.0
MX
MY
MY
0.75
0.5
MX
0.25
0.5 × 10⁻²
DEVX
DEVY
DEVX
0.1 × 10⁻²
DEVY
0
1000
2000
3000
4000
5000
→ N (Number of steps)

Fig. 4.4

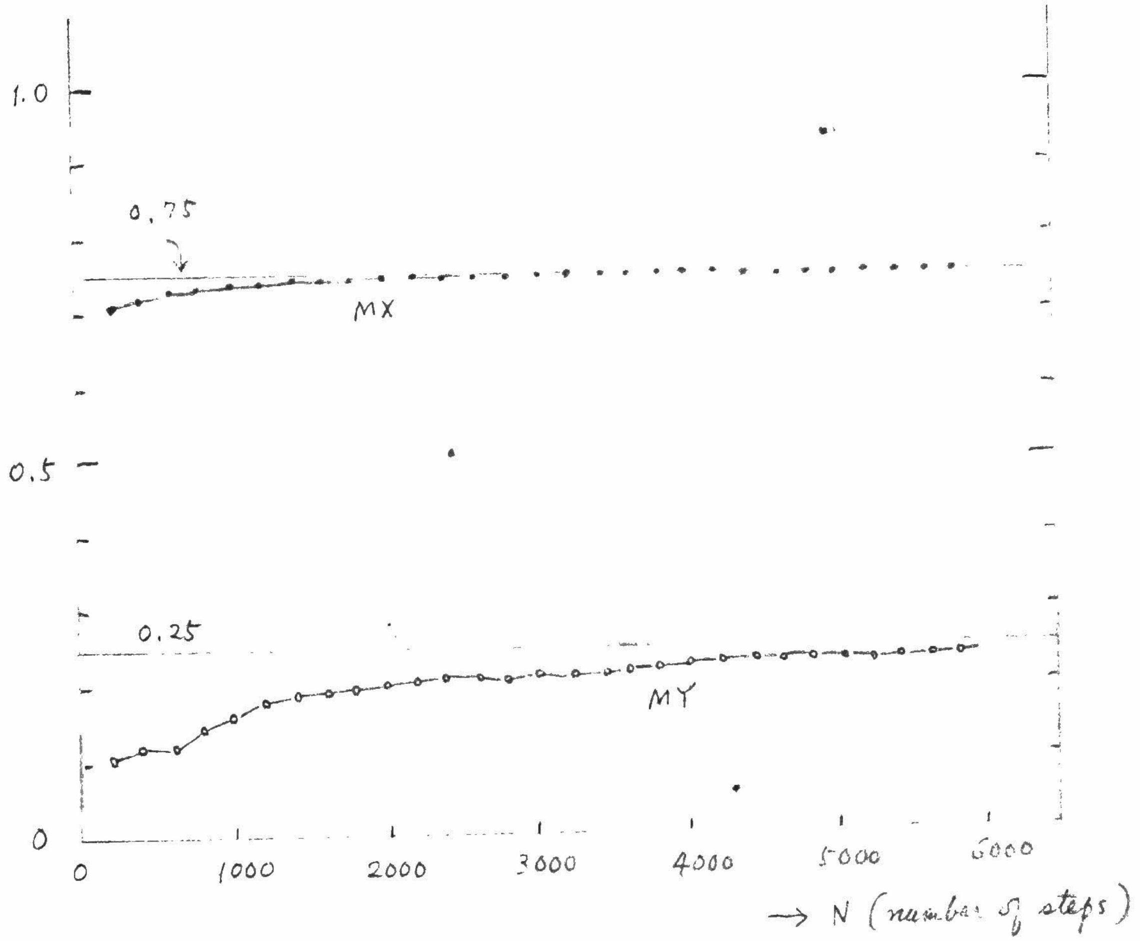

1.0
0.75
MX
0.5
0.25
MY
0
0
1000
2000
3000
4000
5000
6000
→ N (number of steps)

Fig. 4.5

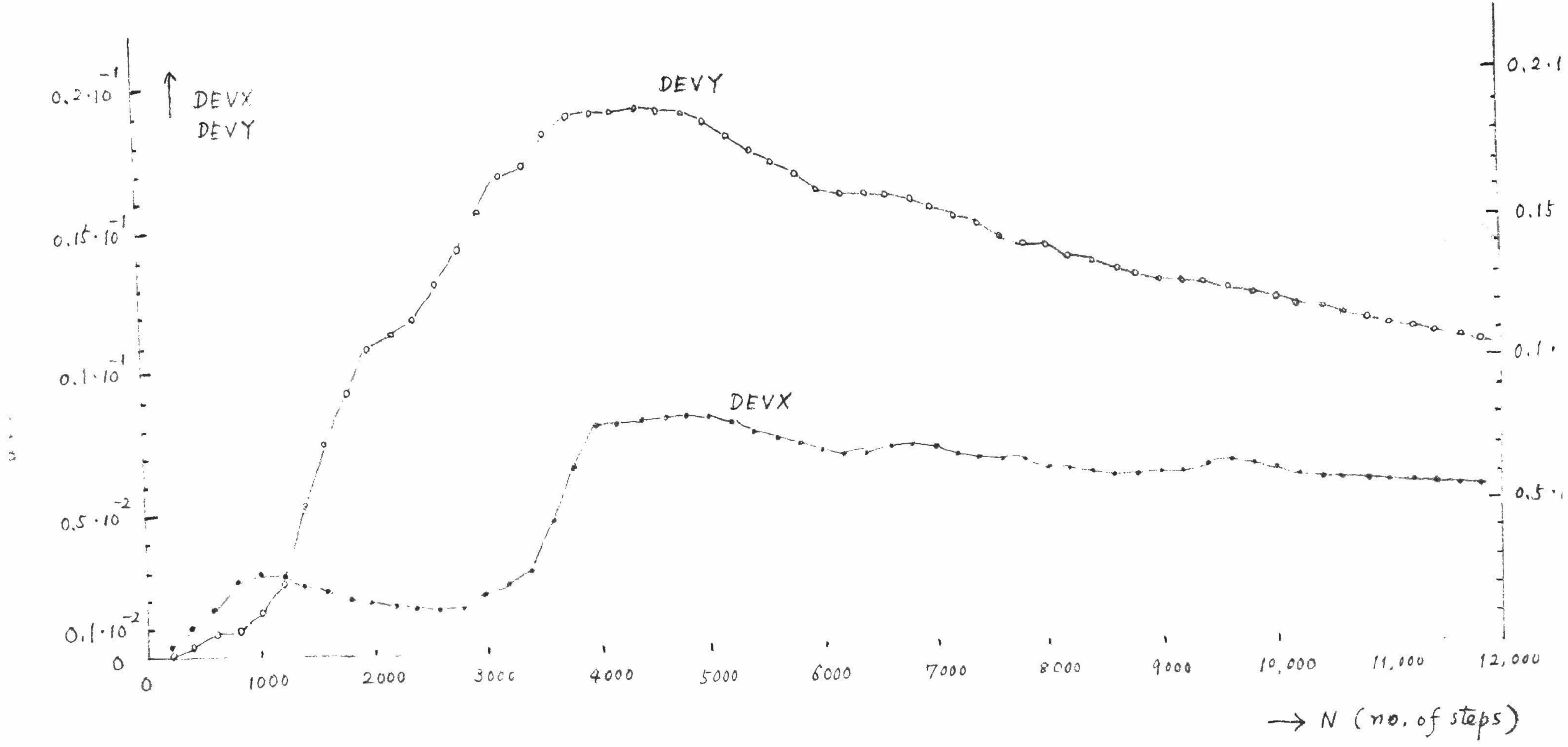

DEVX
DEVY
DEVY
DEVX
0.2·10^-1
0.15·10^-1
0.1·10^-1
0.5·10^-2
0.1·10^-2
0
0
1000
2000
3000
4000
5000
6000
7000
8000
9000
10,000
11,000
12,000
0.2·1
0.15
0.1·
0.5·
→ N (no. of steps)

Fig. 4.7

in Tables 4.2 and 4.3 and Figs. 4.4 and 4.5. After this stage of computation, it was found that the results were satisfactory for the present purpose and the computation was ended.

The reason why a_i^2 is diminished by a factor of 10 in the procedure (41^o) is clear in Figs. 4.6 and 4.7. Here the computation was performed with condition of redefining a_i's deleted from the procedure (41^o). This is a good example to show how effective the increasing of a_i's is for making the potential gradient steep enough. T_m may be fixed throughout the computation. The refining of T_m in the procedure does not seem to be essential.

§4.5 Further extensions

Experimental results so far obtained have confirmed the validity of the theoretical expectation. Further extensions may be accomplished for the following problems:

(a) Conditional minimization:

Find the points $(x_1, x_2, \ldots, x_n)$ where a given function

$$F(x_1, x_2, \ldots, x_n) \quad (>0)$$

assumes the minimum values, under the given conditions that

$$G_i(x_1, x_2, \ldots, x_n) = 0, \ (i=1, 2, \ldots, m).$$

For such a type of problems, we may as well introduce a pseudo-potential

$$U(x_1, x_2, \ldots, x_n) = a^2 F^2 + \sum_{1}^{m} a_i^2 G_i^2 ,$$

$$(a \text{ and } a_i\text{'s} = \text{constants}),$$

then the potential minimum will give the solution. Maximization problems can be treated in the same manner.

(b) Complex roots of simultaneous nonlear algebraic equations:

Transforming the variables in Eq. (4.1) by the relation

$$x_k = \alpha_k + j\beta_k \quad (k = 1, 2, \cdots, n ;$$
$$j^2 = -1 ; \alpha_k , \beta_k : \text{real}),$$

we have simultaneous nonlinear algebraic equations with 2n unknowns α_k, β_k (k=1, ... , n). Therefore, at the cost of the increase of the number of unknowns, the method so far explained will also be applicable and the complex roots of Eq. (4.1) will be found approximately.

§4.6 Acknowledgement

The authors are very grateful to Prof. H. Nishihara for his helpful discussions. All the computation done for the confirmation of the present Monte Carlo method was performed on KDC-I.

REFERENCES (Chapter 4)

[1] Todd, J. (editor): Survey of Numerical Analysis. New York: McGraw Hill 1962.

[2] Landau, L. D. and E. M. Lifshitz: Statistical Physics. Reading, Mass. : Addison-Wesley 1958.

PART 4*)

THEORETICAL INTERPRETATION OF THE EQUATORIAL SPORADIC E LAYERS

--- AN INSTABILITY ANALYSIS OF A WEAKLY-IONIZED PLASMA IN AN ELECTROMAGNETIC FIELD ---

*) This work was done in collaboration with Prof. K. Maeda of the Department of Electronics, and Prof. H. Maeda of the Geophysical Institute, of Kyoto University. Part of the material was presented at the International Conference on Equatorial Aeronomy, held in Lima, Peru, in September, 1962. Revised version of the paper was read in the 14th General Assembly of URSI, held in Tokyo, in September 1963.

Chapter 5. Theoretical interpretation of the equatorial sporadic E layers

In this chapter a macroscopic theory is presented in order to interpret the anomalously large foEs observed near the magnetic equator. A. Simon's method is modified and applied to the ionosphere and it was found that when the W-E electric field strength exceeds a certain threshold, an inhomogenetiy in charge density occurs. Since the W-E electric field distribution is very similar to the W-E electric current, it can be understood that the instability leading to the formation of the inhomogeneity is well correlated with the electrojet. The source of sporadic E reflection is this inhomogeneity; but the mechanism involved is not identical to that of the two-stream instability.

In order to make comparison between the theory and the phenomena, the variation of horizontal magnetic force observed at various stations in 1958 is analyzed and the meridional distribution and diurnal variation of jet current and of electric field are calculated. The results of calculations are compared with the diurnal variation of foEs at Huancayo and with the dip dependence of noon foEs obtained from observations in 1958.

§5.1 Introduction

The daytime foEs near the magnetic (dip) equator is abnormally large and this zone of intense Es is very narrow (about $\pm$ 5° in dip), centered on the dip equator [Matsushita, 1951; 1952a, b; 1953a, b]. Matsushita also showed that the foEs distribution against the dip resembles closely the distribution of the daily range in horizontal magnetic force and concluded that there must be close relationship between the equatorial Es (Es(q)) and the electrojet.

As for the structure and orientation of Es(q), a number of important findings have come out from the high power radar experiment [Bowles et al., 1957, 1960, 1963; Cohen et al., 1963].

Up to this day, some authors have attempted to describe the formation of the equatorial Es, e. g. from the view-point of two-stream plasma wave micro-instability [Farley, 1963].

In this paper the authors, through a macroscopic approach, try to explain theoretically the formation and orientation of a certain inhomogeneity in the E region and to interpret several features of Es(q) on the basis of the theoretical results. The mechanism involved is to be distinguished from the so-called two-stream instability, since the gradient of the background charge distribution plays an essential part.

§5.2 Outline of the theory

Our theoretical basis is similar to that worked out by A. Simon [1963], which is aimed at the instability analysis of weakly-ionized plasmas. Particular modifications are made so as to make Simon's method applicable to the present ionospheric model.

Let the rectangular coordinates be x, y, and z axes, which are in the geomagnetic N-S direction, W-E direction, and vertical upward direction, respectively (see Fig. 5.1). The background electron density (=positive ion density) is assumed to show a linear variation over a height range of L. The electrostatic force E_0 (> 0) is applied in the W-E direction. Let superscripts "+" and "-" be attached to the quantities belonging to the positive ions and electrons respectively. Suffix "$\perp$"("$\parallel$") is used to show that the coefficient referred to is that perpendicular (parallel) to the magnetic field. μ , D, ω_H, and τ_c are defined to be the mobility, the diffusion coefficient, the angular gyration frequency, and the mean collision time, respectively. If we consider a partially-ionized medium, with an electric field $\vec{E}$ (E_x, E_y, E_z), then the fluxes of the ions and electrons, $\vec{j}^{\pm}$, are given by

$$j_x^{\pm} = -D_{\parallel}^{\pm} \frac{\partial n}{\partial x} \pm \mu_{\parallel}^{\pm} n E_x , \tag{5.1}$$

$$j_y^{\pm} = -D_{\perp}^{\pm} \frac{\partial n}{\partial y} \pm \mu_{\perp}^{\pm} n E_y \mp (\omega_H \tau_c)^{\pm} \left(-D_{\perp}^{\pm} \frac{\partial n}{\partial z} \pm \mu_{\perp}^{\pm} n E_z \right), \tag{5.2}$$

$$j_z^{\pm} = -D_{\perp}^{\pm} \frac{\partial n}{\partial z} \pm \mu_{\perp}^{\pm} n E_z \pm (\omega_H \tau_c)^{\pm} \left(-D_{\perp}^{\pm} \frac{\partial n}{\partial y} \pm \mu_{\perp}^{\pm} n E_y \right). \tag{5.3}$$

The charge density pattern made up from plane waves with wave front parallel to the z-axis and with wave vector lying in the

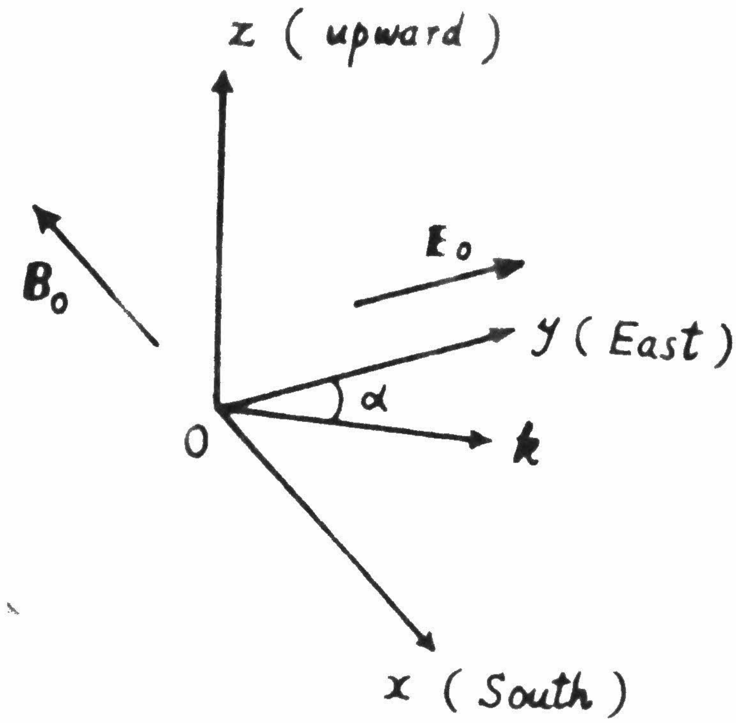


Fig. 5 1

xy-plane (i. e. horizontal plane) is considered. Therefore we put

$$n = n_0(z) + n_1(z) \exp\left(-i\omega t + ik_x x + ik_y y\right), \qquad (5.4)$$

$$\vec{E} = \vec{E}_0(z) - \mathrm{grad}\left\{V_1(z)\exp\left(-i\omega t + ik_x x + ik_y y\right)\right\}, \qquad (5.5)$$

where n_0 and $\vec{E}_0$ $(0, E_0, 0)$ $(E_0 > 0)$ are the unperturbed parts, here assumed to be linear functions of z, while the second terms in the right-hand side of the equations above are the perturbation terms. Continuity equations of particle fluxes of both species

$$\frac{\partial n}{\partial t} = -\mathrm{div}\,\vec{j}^{\pm}, \qquad (5.6)$$

together with Eq. (5.1) through (5.3), yield the following relation:

$$\begin{aligned} 0 = \frac{\partial n}{\partial t} &+ \frac{\partial}{\partial x}\left[-D_{\parallel}^{\pm}\frac{\partial n}{\partial x} \pm \mu_{\parallel}^{\pm} n E_x\right] \\ &+ \frac{\partial}{\partial y}\left[-D_{\perp}^{\pm}\frac{\partial n}{\partial y} \pm \mu_{\perp}^{\pm} n E_y \mp (\omega_H\tau_c)^{\pm}\left(-D_{\perp}^{\pm}\frac{\partial n}{\partial z} \pm \mu_{\perp}^{\pm} n E_z\right)\right] \\ &+ \frac{\partial}{\partial z}\left[-D_{\perp}^{\pm}\frac{\partial n}{\partial z} \pm \mu_{\perp}^{\pm} n E_z \pm (\omega_H\tau_c)^{\pm}\left(-D_{\perp}^{\pm}\frac{\partial n}{\partial y} \pm \mu_{\perp}^{\pm} n E_y\right)\right]. \end{aligned} \qquad (5.7)$$

The above equations are then linearized, after substituting Eqs. (5.4) and (5.5) in Eq. (5.7), i. e.

$$
\begin{aligned}
0 = {} & \left[-i\omega + D_{\parallel}^{\pm} k_x^2 + D_{\perp}^{\pm} k_y^2 \pm \mu_{\perp}^{\pm} i k_y E_0(z)\right] n_1(z) \\
& + \left[\pm \mu_{\parallel}^{\pm} k_x^2 n_0(z) \pm \mu_{\perp}^{\pm} k_y^2 n_0(z)\right] V_1(z) \\
& - D_{\perp}^{\pm} \frac{d^2 n_1}{dz^2} + (\omega_H \tau_c)^{\pm} \mu_{\perp}^{\pm} n_0(z)(i k_y) \frac{dV_1}{dz} \\
& \mp \mu_{\perp}^{\pm} \frac{d}{dz}\left(n_0(z) \frac{dV_1}{dz}\right) \\
& + (\omega_H \tau_c)^{\pm} \mu_{\perp}^{\pm} \left[-i k_y \frac{d}{dz}\left(n_0(z) V_1(z)\right)\right. \\
& \qquad \left. + \frac{d}{dz}\left(n_1(z) E_0(z)\right)\right].
\end{aligned}
\tag{5.8}
$$

If it is assumed that the perturbation is of the form

$$
n_1(z) = \mathcal{N} \sin\left(\frac{\pi z}{L}\right),
$$

$$V_1(z) = \mathcal{V} \sin\left(\frac{\pi z}{L}\right), \qquad (5.9)$$

($\mathcal{N}$, $\mathcal{V}$ = constants of either sign),

then multiplying both sides of Eq. (5.7) by $\sin(\pi z/L)$ and integrating over z from zero up to L, we have the following equations:

$$0 = \left[-i\omega + D_\perp^\pm K_\pm(\alpha) k_y^2 \pm \mu_\perp^\pm (ik_y)\overline{E_0} + (\omega_H\tau_c)^\pm \mu_\perp^\pm \overline{\frac{dE_0}{dz}}\right]\mathcal{N}$$

$$+\left[\pm\mu_\perp^\pm K_\pm(\alpha)(ik_y)\overline{n_0} + (\omega_H\tau_c)^\pm \mu_\perp^\pm \overline{\frac{dn_0}{dz}}\right](-ik_y\mathcal{V}), \qquad (5.10)$$

where the overbar is such that

$$\frac{2}{L}\int_0^L f(z)\sin^2\left(\frac{\pi z}{L}\right)dz \equiv \overline{f(z)} \qquad (5.11)$$

for any function f(z). We have used simplified notations

$$K_\pm(\alpha) \equiv \frac{D_\parallel^\pm}{D_\perp^\pm}\tan^2\alpha + 1 \cong \frac{\mu_\parallel^\pm}{\mu_\perp^\pm}\tan^2\alpha + 1, \qquad (5.12)$$

where

$$\tan\alpha = k_x/k_y \qquad \text{(see Fig. 5.1).} \qquad (5.13)$$

In deriving Eq. (5.10), we have assumed that

$$k_y^2 \gg (\pi/L)^2,$$

i. e.

(a) $L \gg k_y^{-1}$. (5.14)

It was also noted that

$$\int_0^L n_0(z)\sin\left(\frac{2\pi z}{L}\right)dz \cong \frac{dn_0}{dz}\int_0^L z\sin\left(\frac{2\pi z}{L}\right)dz \cong \mp\overline{\frac{dn_0}{dz}}\cdot\frac{L^2}{2\pi},$$

since $n_0(z)$ is understood to be a linear function of z. In order now that we have nontrivial solution for Eq. (5.10), the determinant of the coefficients must be zero. Hence we have an algebraic equation, which we solve for $i\omega$. In order to determine the conditions for stability, we need consider only the real part of $(-i\omega)$. Thus we have

$$R_e(-i\omega) = \frac{\frac{3}{2}(\mu_\perp^+ + \mu_\perp^-)\left[\{(\omega_H\tau_c)^+ + (\omega_H\tau_c)^-\}\mu_\perp^+\mu_\perp^-\overline{E_0}\left(\frac{1}{\overline{n_0}}\overline{\frac{dn_0}{dz}}\right) - (\mu_\perp^+ D_\perp^- + \mu_\perp^- D_\perp^+)K_+(\alpha)K_-(\alpha)k_y^2\right]}{\frac{9}{4}\{(\omega_H\tau_c)^+\mu_\perp^+ - (\omega_H\tau_c)^-\mu_\perp^-\}^2\left(\frac{1}{\overline{n_0}}\overline{\frac{dn_0}{dz}}\right)^2 k_y^{-2} + (\mu_\perp^+ K_+(\alpha) + \mu_\perp^- K_-(\alpha))^2},$$

(5.15)

where additional assumptions, i. e.

(b) $\left(\frac{1}{\overline{n_0}}\overline{\frac{dn_0}{dz}}\right)^2 \ll k_y^2$, (5.16)

(c) $\left|\frac{1}{\overline{n_0}}\overline{\frac{dn_0}{dz}}\right| \gg \left|\frac{1}{\overline{E_0}}\overline{\frac{dE_0}{dz}}\right|$, (5.17)

and

(d) $1 \gg \tan^2 \alpha$ (5.18)

were utilized. Assumptions (a) and (b) are evidently permissible, while (c) states that the electrostatic field does not change so remarkably as the charge density in the vertical direction in the equatorial E layer. Assumption (d) reflects that we are mainly interested in the case of longitudinal waves which propagate nearly perpendicular to the geomagnetic field.

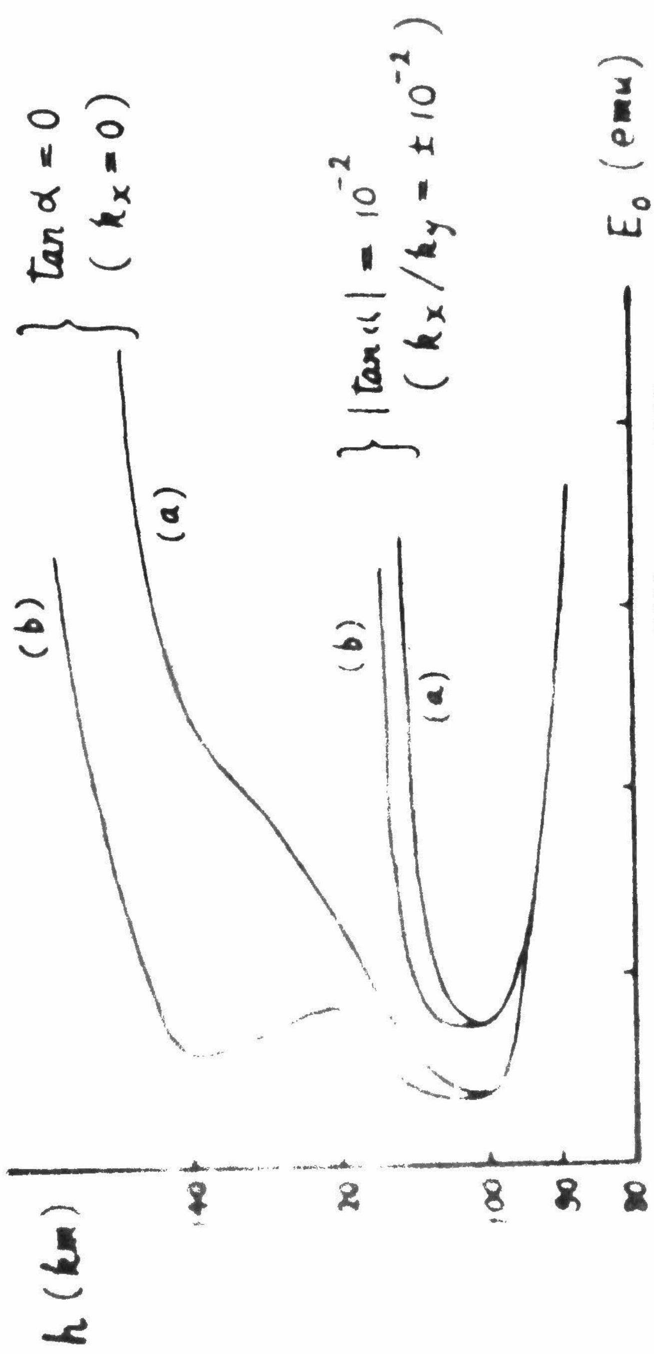
From Eq. (5.15), in order for the growth condition ($R_e(-i\omega) > 0$) to be satisfied, it is at least required that

$$\bar{E}_0 \cdot \left(\frac{1}{n_0} \cdot \overline{\frac{dn_0}{dz}} \right) > 0 . \qquad (5.19)$$

When $E_0 > 0$, as is the case in daytime at the magnetic equator, positive gradient (vs. height) of background charge density is required. This condition is realized from below the normal E layer towards higher levels, where there can exist back-scattering blobs produced from growing instabilities.

The physical mechanism of this type of instability is that, in case of background charges increasing vs. height, the horizontal polarization electric fields accompanying the charge density pattern intensify the trends of the perturbation by help of the drifting motion of charges in the vertical direction.

From Eq. (5.15), the magnitude of the electrostatic field E_0 necessary for the instability to grow is calculated (the inhomogeneity wavelength = 100 m) and shown in Fig. 5.2, for the two cases $\alpha = 0$ and $\alpha \neq 0$. Slightest departure from $\alpha = 0$ corresponds to a very high threshold electric field. In the case (a) of the



h (km)
140
120
100
90
80
0
1000
2000
3000
4000
E₀ (emu)
(a)
(b)
tan α = 0
(k_x = 0)
|tan α| = 10^-2
(k_x / k_y = ± 10^-2)

figure, $\left(\frac{1}{\bar{n}_0}\cdot\frac{\overline{dn_0}}{dz}\right)^{-1}$ is assumed to vary proportionally with the scale height. The value of $\left(\frac{1}{\bar{n}_0}\cdot\frac{\overline{dn_0}}{dz}\right)^{-1}$ at the altitude of 100 km is taken to be 1 km. The case (b) corresponds to the case in which $\left(\frac{1}{\bar{n}_0}\cdot\frac{\overline{dn_0}}{dz}\right)^{-1}$ is assumed to be constant (=1 km) vs. height. It is believed that these assumptions are not far from reality in the lower region of the E layer where the gradient of the height distribution of charge density is very outstanding. In the Fig. 5.2 the minimum value of the threshold E_0 is approximately 300 emu at the altitude of 100 km when $\alpha=0$. It is also seen from the same figure that when E_0 exceeds 700 emu, then the instability will grow in the height range of 96 ~ 113 km in the case (a) ($\alpha=0$), and 96 ~ 118 km in the case (b) ($\alpha=0$). In the latter case the instability may occur in the region still higher than the one mentioned.

In short we may conclude that a plane-wave inhomogeneity can hardly occur unless the direction of its wave vector coincides with that of the y-axis (W-E direction); and the shape of the inhomogeneity is like a thin plate hanging down vertically with its surface lying in the xz-plane. The geometrical structure of the inhomogeneity is such that the width in the N-S direction coincides with that of the electrojet (see Fig. 5.8), the height range is restricted by the threshold condition of Fig. 5.2, and as for the spacing in the W-E direction, our linear theory has not much to tell, although the mean irregularity wavelength may take on values from a few meters to a few hundred meters. This inhomogeneity is believed to give rise to the enhanced radio-wave back-scattering, commonly called Es(q).

If we use the value of $\left(\frac{1}{\overline{n_0}} \cdot \frac{\overline{dn_0}}{dZ}\right)^{-1}$ twice as large as that we have used in Fig. 5.2, then the threshold electric field E_0 will become twice larger than that indicated in Fig. 5.2. Even in this situation, the general features so far duscussed are not altered.

§5.3 Observed results and applications of the theory

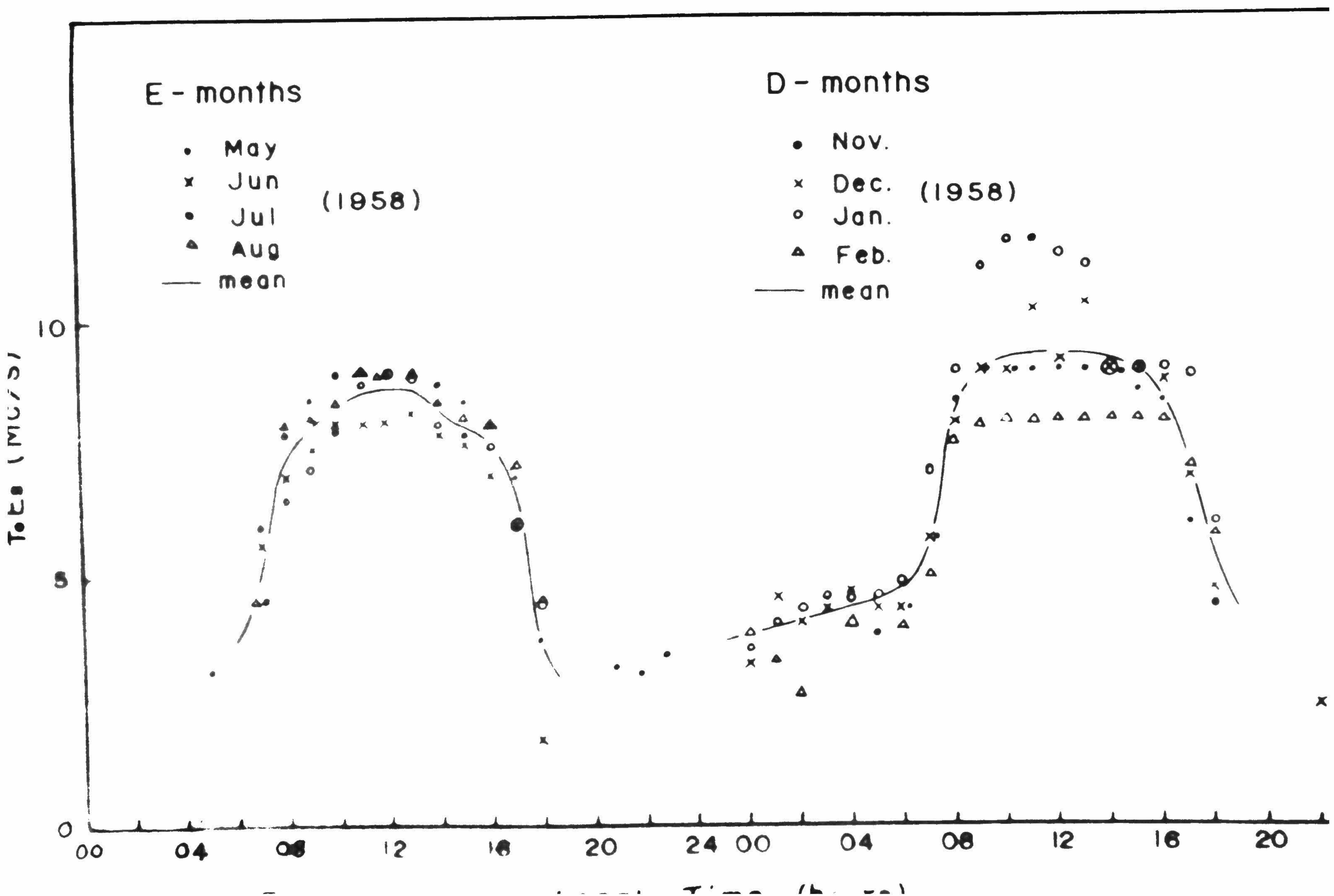
5.3.1 Observed results of Es and geomagnetic variation

A number of investigators have already reported on the dip dependence and diurnal variation of foEs. We have investigated some similar characteristics of foEs, using the data of the stations listed in Table 5.1, on 5 international quiet days for each month in 1958. Fig. 5.3 shows the diurnal variations of foEs at Huancayo, which are the average of foEs on twenty quiet days in four months for two seasons. May, June, July and August are grouped in J-months; March, April, September and October in E-months; November, December, January, and February in D-months. It can be seen from the figure that foEs rises rapidly around 07 hr and falls around 17 hr. In Fig. 5.4 foEs is plotted against the dip of the station, where foEs is the average of the values observed at noon on twenty quiet days for each season (J, E and D-months). As seen from the smoothed curves foEs is very large over the range of $\pm 5^{\circ}$ in dip angle. No seasonal change is distinguishable in the shape and width of high foEs zone.

Next Fig. 5.5 presents the diurnal variation of the external part of the horizontal magnetic force (ΔH^{e}) at Huancayo averaged

Table 1 List of Ionospheric and Geomagnetic Stations

No.	Ionospheric Station	Dip (°)	Geographic Lat. (°)	Geographic Long. (°)	Geomagnetic Station	Dip (°)	Geographic Lat. (°)	Geographic Long. (°)
1	Armedabad	34	23.0 N	73.6 E	Helwan	41.2	29.9 N	31.3 E
2	Haringhata	30	22.9 N	88.6 E	Honolulu	39.0	21.3 N	158.1 W
3	Macau	30			Cha-Pa	29.5	22.4 N	103.8 [illegible]
4	Bombay	24.6	18.6 N	72.8 E	Alibag	24.6	18.6 N	72.9 E
5	Baguio	19	16.4 N	120.6 E	M'Bour	17.4	14.4 N	17.0 W
6	Talara	12.5	4.6 S	81.3 W	Muntinlupa	14.2	14.4 N	121.0 E
7	Madras	10.5	13.0 N	80.3 E	Guam	12.9	13.5 N	144.8
8	Chiclayo	9.7	6.8 S	79.8 W	Talara	12.5	4.6 S	81.3 W
9	Chimbote	6.3	9.1 S	78.4 W	Fanning	10.5	3.9 N	159.4 W
10	Djibouti	5.6	11.5 N	43.9 E	Chiclayo	9.7	6.8 S	79.8 W
11	Tiruchirapa-lli	4.8	10.8 N	78.8 E	Chimbote	6.3	9.1 S	78.4 W
12	Kodaikanal	3.4	10.2 N	77.5 E	Chidambaram	5.4	11.4 N	79.7
13	Huancayo	1.9	[illegible].1 S	[illegible].3 W	Kodaikanal	3.4	10.2 N	77.5 E
14	Trivandrum	-0.6	8.5 N	77.0 E	Jarvis	[illegible].2	0.4 N	160.0 W
15	Natal	-1	5.3 S	35.1 W	Huancayo	1.9	12.1 S	75.3 W
16	La Paz	-[illegible].7	16.[illegible] S	6[illegible].0 W	Koror	-0.1	7.3 N	134.5 E
17	Ibadan	-6	7.[illegible] N	[illegible]4.0 E	Trivandrum	- .6	8.5 N	77.0 E
18	Bangui	-13.[illegible]	4.4 N	18.6 E	Addis Ababa	-0.9	9.0 N	38.8 E
19	Singapore	-17	1.3 N	103.8 E	Yanca	-4.4	15.5 S	74.5 W
20	Hollandia	-20	2.6 S	14[illegible].5 E	Bangui	-13.9	4.4 N	18.6 E
21	Sao Paulo	-22	23.6 S	46.6 W	Hollandia	-20.8	2.6 S	140.5
22	Lwiro	-30	2.3 S	28.8 E	Tahiti	-29,7	17.6 S	149.6 W
23					Apia	-30.2	13.8 S	171.8 W
24					Djakarta	-[illegible]2.4	6.0 S	106.7 E
25					Luanda	-41.7	8.9 S	13.1 E



E - months
May
Jun
Jul
Aug
mean
(1958)
D - months
Nov.
Dec.
Jan.
Feb.
mean
(1958)
10
5
0
00
04
08
12
16
20
24 00
04
08
12
16
20

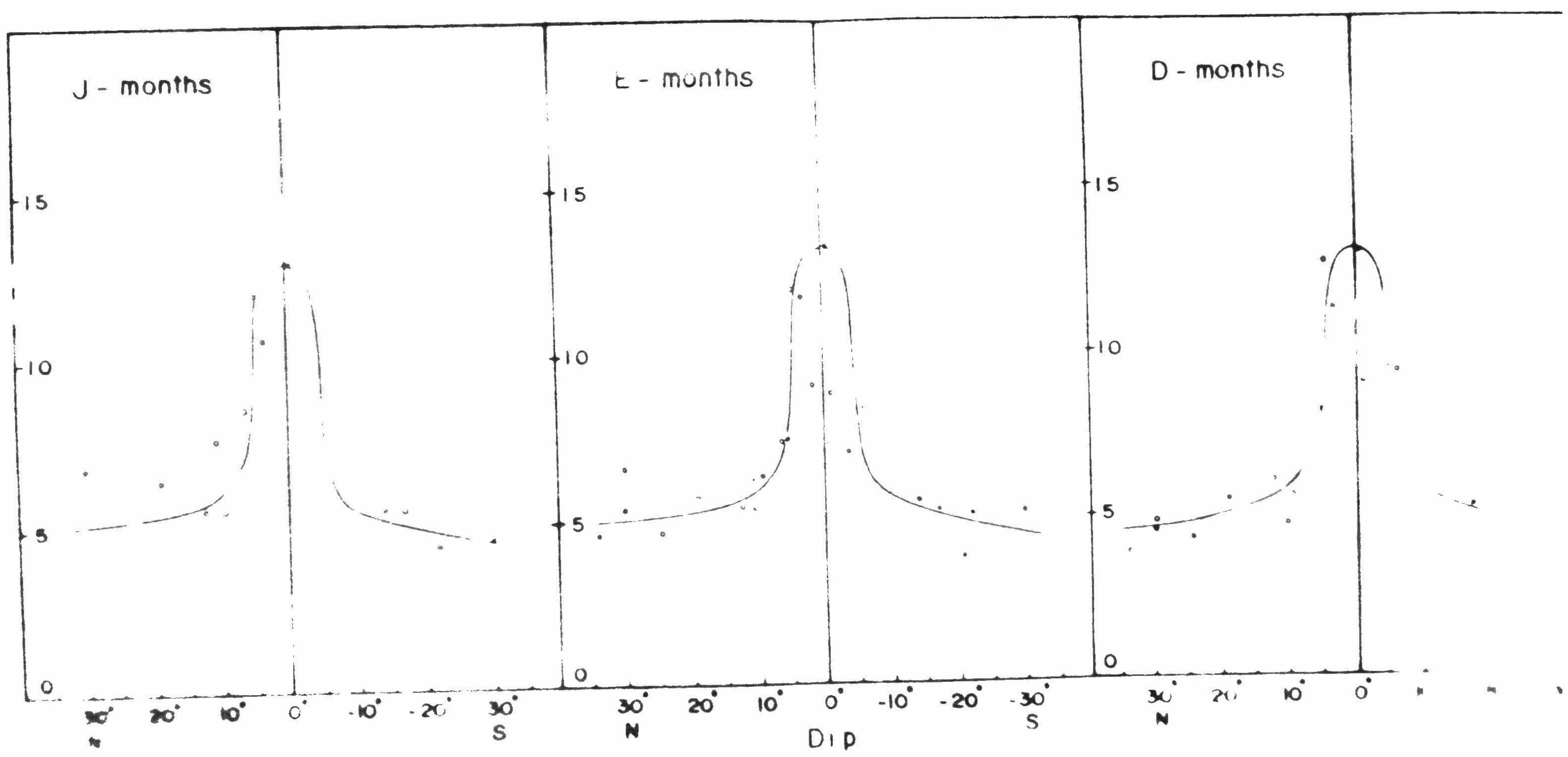


Fig. 5.4

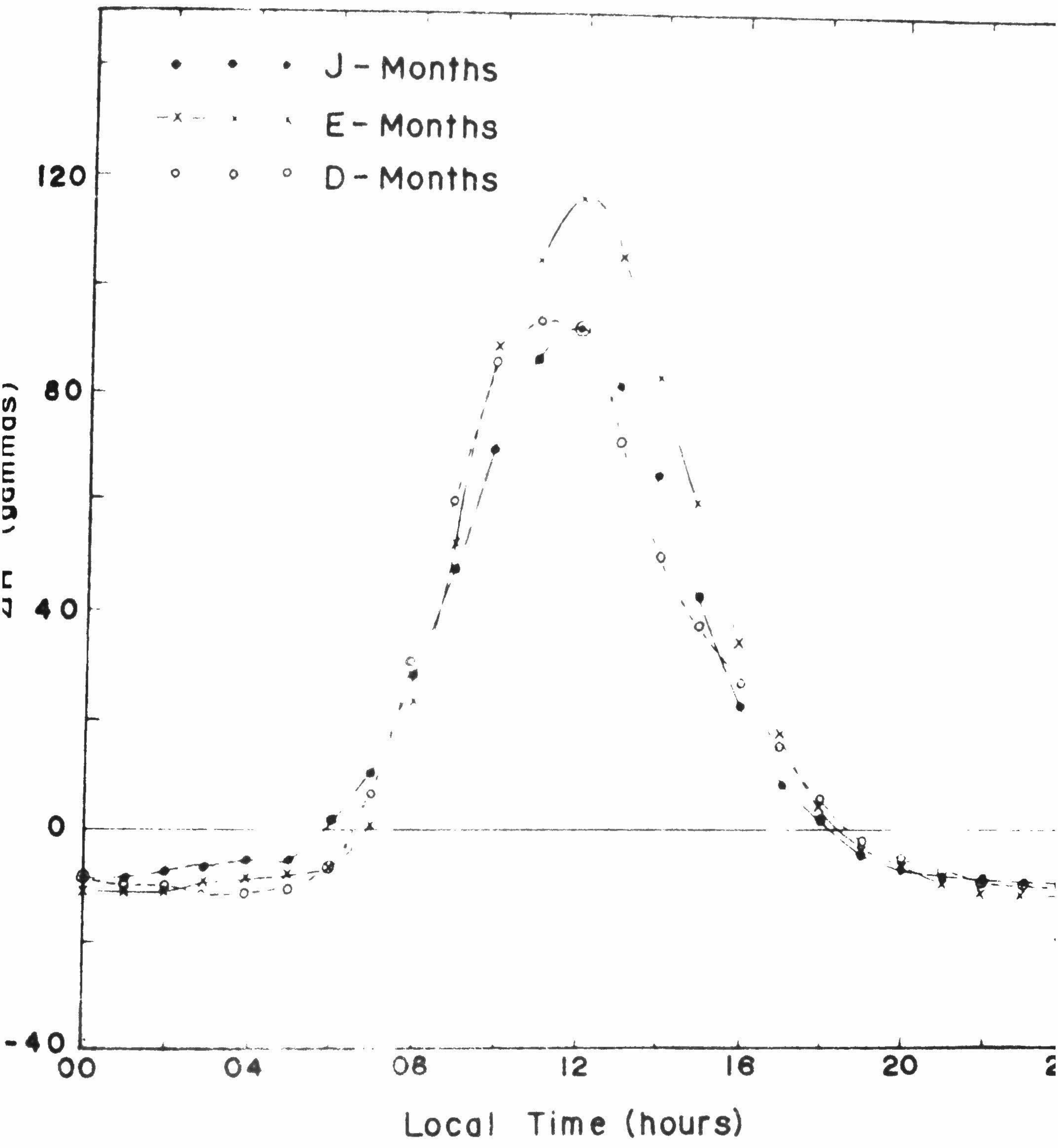

J - Months
E - Months
D - Months
120
80
40
0
-40
00
04
08
12
16
20
Local Time (hours)

Fig. 5.5

on twenty quiet days for each season. The zero level in the figure was determined by a method previously found by one of the authors [Maeda H., 1955]. Fig. 5.6 is the similar plot of ΔH^e at noon against the dip of the station as in Fig. 5.4. The stations are listed in Table 5.1. ΔH^e presented here is taken 2/3 of the original values, in order to eliminate the internal part.

5.3.2 Estimation of the electrojet current and the electric field

Suppose the current density i(x) is flowing eastward (y) at the height of h, as shown in Fig. 5.7. Here x is the axis from north to south and measured along a geomagnetic meridian from the dip equator. The earth's curvature ignored. This current gives rise to the horizontal magnetic force (ΔH^e) on the ground. The meridional distance of the ground station from the dip equator is denoted by x'. Then Biot-Savart's law gives

$$\Delta H^e(x') = \int_{-b}^{b} \frac{2h\,i(x)}{h^2 + (x - x')^2}\,dx, \qquad (5.20)$$

where b is the half-width of the current sheet. The integral equation (5.20) is approximated by the following simultaneous equation.

$$\Delta H^e(x_j') = \sum_i \frac{2h \cdot \Delta x}{h^2 + (x_i - x_j')^2} \cdot i(x_i). \qquad (5.21)$$

We take the smoothed curve of ΔH^e as shown in Fig. 5.6, and

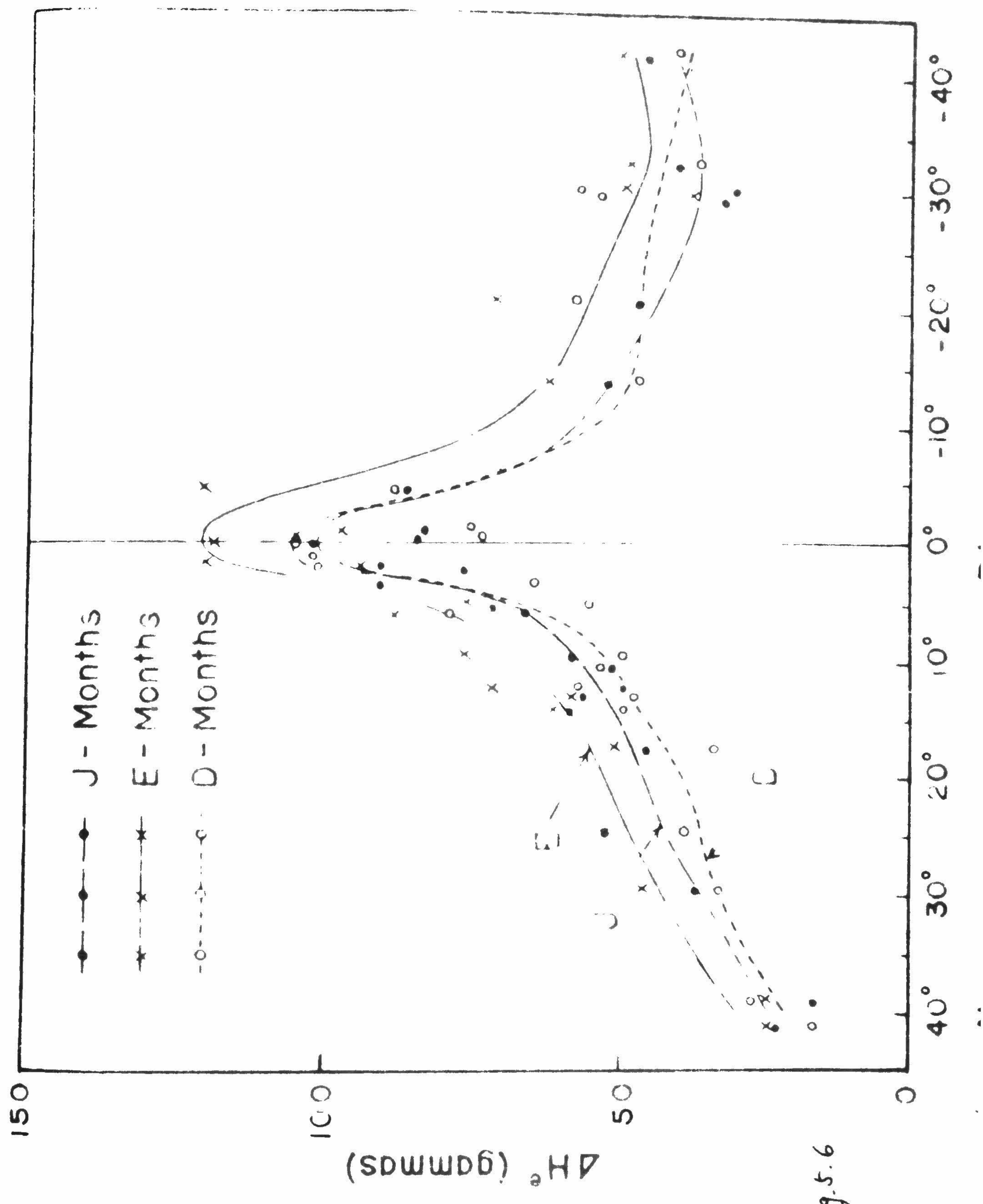
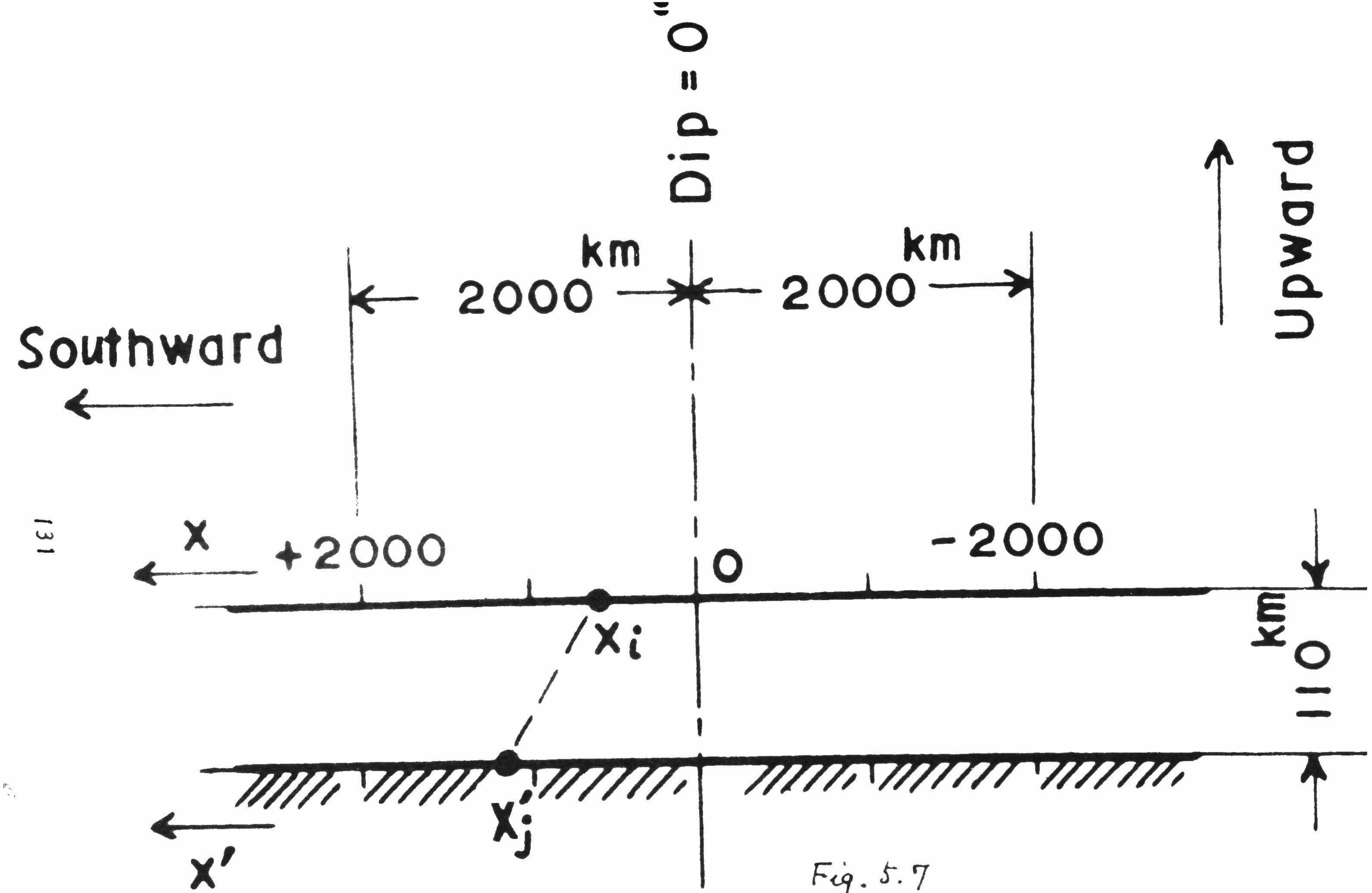

J - Months
E - Months
D - Months
ΔH^e (gammas)
150
100
50
0
40°
30°
20°
10°
0°
-10°
-20°
-30°
-40°
J
E
D

Fig. 5.6



Dip = 0"
Southward
Upward
2000 km
2000 km
x
+2000
0
-2000
110 km
x_i
x'_j
x'

Fig. 5.7

$b=2,00$ km,

$x=100$ km,

x_j', $x_i=-2,000, -1,900, \cdots, -100, 0, 100, \cdots, 2,000$ km,

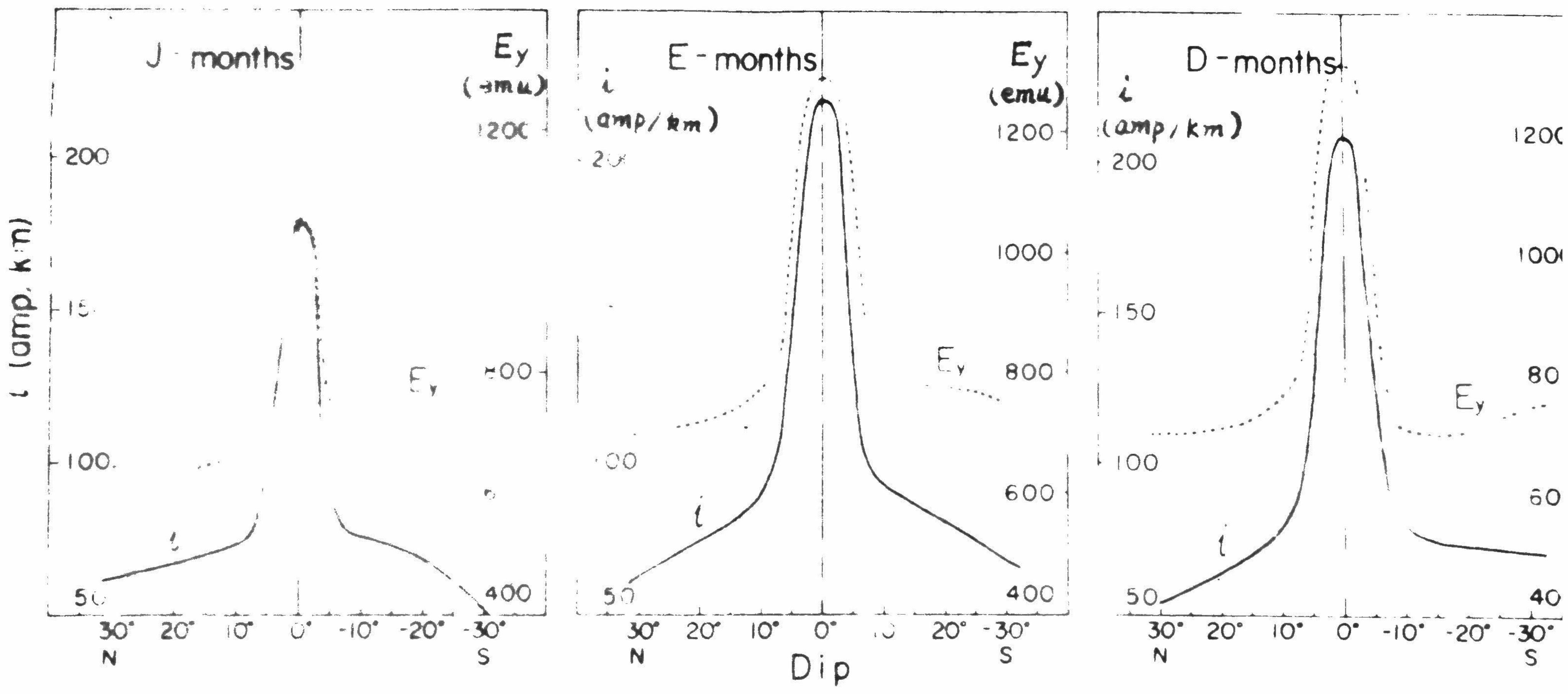
$h=110$ km.

Then the simultaneous equation has 41 unknowns ($i(x_i)$) and can be solved, by using the value $\Delta H^e(x_j')$ read from the smoothed curve in Fig. 5.6. The calculated results of $i(x)$ at noon for each season are shown in Fig. 8. $i(x)$ at other times (10 hr, 14 hr etc.) has been also obtained and using these results, the diurnal variation of i at Huancayo can be obtained, which is shown in Fig. 5.9.

Next in order to obtain the electric field ($E_y=E_o$), which causes the enhanced flow of the jet current, the total conductivity Σ_y , which is the integral of σ_y over the height range of the E region, must be evaluated. As for Σ_y on the dip equator and its neighboring zone, the results of the previous papers [Maeda H., 1953, 1955; Maeda K. and Matsumoto, 1962] are available. Using these results, together with the jet current, we have calculated the electric field ($E_y=E_o$) by the following equation:

$$E_y = i(x) / \Sigma_y \equiv E_o \qquad (5.22)$$

The diurnal variation of E_y ($=E_o$) at Huancayo is shown in Fig. 5.9. In Fig. 5.8 the dip dependence of E_o at noon for each season is shown.



J - months
E - months
D - months
E_y (emu)
i (amp/km)
i (amp. km)
200
100
50
1200
1000
800
600
400
30° N
20°
10°
0°
-10°
-20°
-30° S
Dip

Fig. 5.8

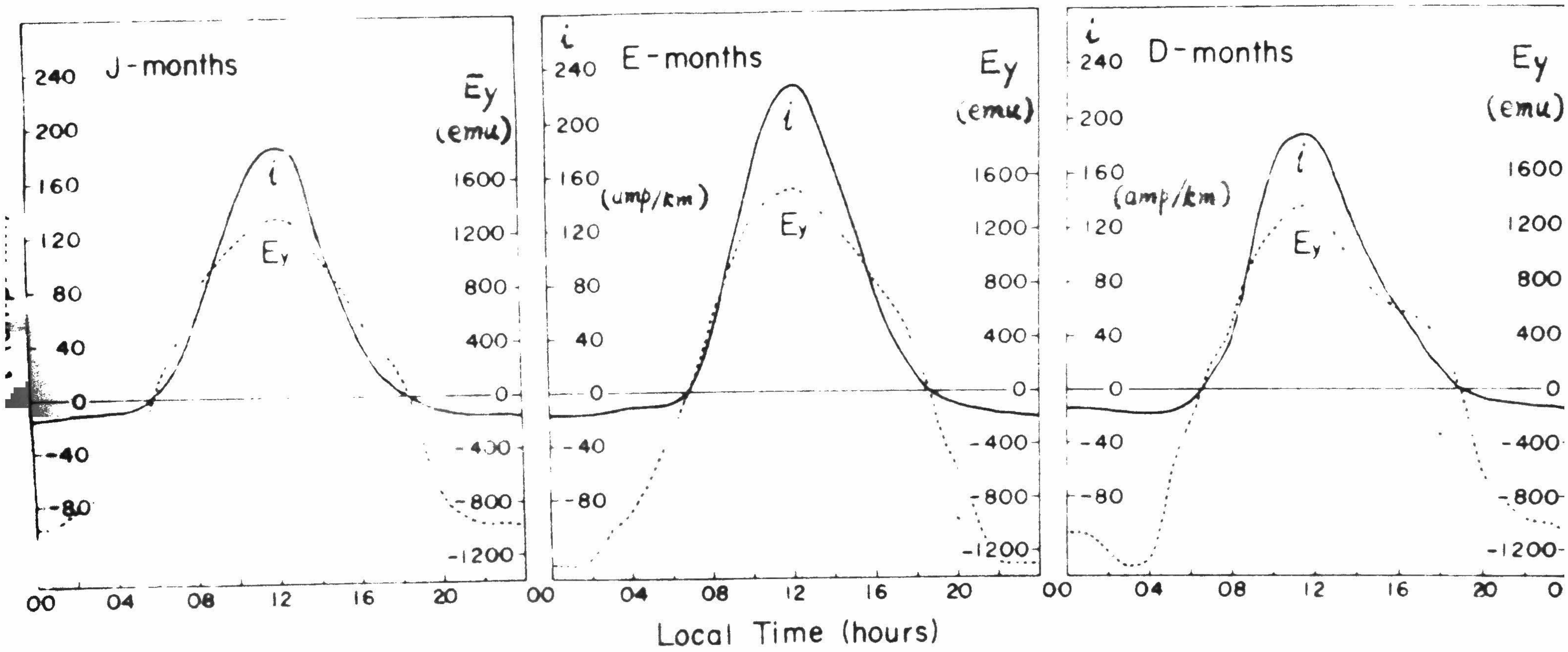

J-months
E-months
D-months
i
E_y
(emu)
(amp/km)
Local Time (hours)
00
04
08
12
16
20
240
200
160
120
80
40
0
-40
-80
1600
1200
800
400
-400
-800
-1200

Fig. 5.9

5.3.3 Comparison of the theory with the observed data

We have already obtained the threshold value of E_0 in Fig. 5.2. Comparing this result with Fig. 5.9 (the diurnal variation of electric field), we see that an inhomogeneity occurs at about 07~08 hr and disappears at about 16 ~17 hr. And from the comparison between Fig. 5.2 and Fig. 5.8, it can also be seen that at local noon an inhomogeneity can occur on the magnetic equatorial zone where the electrojet flows. Outside this narrow zone, E_0 is small and does not reach the threshold value.

§5.4 Discussions and remarks

The suggestion first made by Matsushita and now widely supported, that the Es(q) is closely correlated with the electrojet, has been demonstrated by our present theory, in terms of the W-E electric field strength. The main point of the theory is that starting from macroscopic equations, where the height-dependent structure of the ionosphere is considered, large W-E electric field is possible to result in an unstable plasma state under the action of the geomagnetic field. In the present theory the electric field does not cause the instability by itself but in cooperation with the magnetic field, whereas in the theory based on the two-stream drift instability [Farley, 1963] the magnetic field does not play any essential part but has merely a secondary effect. We may conclude that the main features concerning the diurnal variation of foEs at Huancayo and the dip dependence of foEs near the dip equatorial zone have been explained.

According to the radar experiments by Bowles et al.[1957, 1960, 1963], the irregularity which gives rise to Es(q), is not of field-aligned cylindrical shape as hitherto supposed, but it is a certain plane wave disturbance, which lies in the plane parallel to the geomagnetic field. This fact is consistent with the structure and orientation given by the present theory. A Doppler shift was observed in the received radio waves scattered back from the irregularity, and the scatterers moved from east to west. From the analysis of the imaginary part of $(-i\omega)$, i. e. $\mathcal{I}m\,(-i\omega)$, for $\alpha = 0$, we have at the altitude of about 100 km

$$\mathcal{I}m(-i\omega)/i \cong \Big[(\omega_H \tau_c)^- \mu_\perp^- D_\perp^+ \mu_\perp^+ \left(\frac{1}{\overline{n}_0} \cdot \frac{\overline{dn_0}}{dz} \right) k_y \frac{3}{2} + \{(\omega_H \tau_c)^-\}^2 \mu_\perp^+ (\mu_\perp^-)^2 E_0 \frac{9}{4} \left(\frac{1}{\overline{n}_0} \frac{\overline{dn_0}}{dz} \right)^2 k_y^{-1} + \text{small terms} \Big]$$

divided by the denominator of Eq. (5.15).

Hence we see that in order for ω to be positive k_y must be negative, which means that the longitudinal waves arising from our instability always travel westward. We also see that, substituting appropriate values for the various physical parameters in the above equation, the phase velocity of the waves ω / k_y is equal to the sound velocity in the order of magnitude.

It will be noticed that the present theory may also be effective in interpreting the auroral type Es, since there is a high electric field with an intense current flow in the auroral zone. This particular type of mechanism, which has so far been involved in the interpretation of Es(q), might provide a powerful means for the understanding of the so-called spread-F phenomena.

The present theory based on Simon's work is a linearized treatment, and therefore we cannot obtain any information on the density excess in an inhomogeneity. We have avoided running the risk of discussing all the features of the phenomena by the growth rates etc. predicted from the simple linearized treatment.

REFERENCES (Chapter 5)

Bowles, K. L. and Cohen, 1957, AST, 41, 1

Bowles, K. L., R. Ochs, and Balsley, 1960, J. Geophys. Res., 65, 1853.

Bowles, K. L., B. Balsley, and R. Cohen, J. Geophys. Res., 68, 4-5.

Cohen, R. and K. L. Bowles, 1963, J. Geophys. Res., 68, 503.

Farley, D. T., 19, Phys. Rev. Letters, 10,

Maeda, H., 1953, J. Geomag. Geoelec., 5, 4

Maeda, H., 1955, Rept. Ionosph. Res. Japan, 9, 148.

Maeda, H., 1956, Rept. Ionosph. Res. Japan, 10, 49.

Maeda, K., and H. Matsumoto, 1962, Rept. Ionosph. Space Res. Japan, 16, 1.

Matsushita, S., 1951, J. Geomag. Geoelec., 3, 44.

Matsushita, S., 1952a, Rept. Ionosph. Res. Japan, 6, 1.

Matsushita, S., 1952b, Rept. Ionosph. Res. Japan, 6, 3.

Matsushita, S., 1953a, Rept. Ionosph. Res. Japan, 7, 45.

Matsushita, S., 1953b, J. Geomag. Geoelec.

Simon, A., Phys. Fluids

Summary

Main points of the results obtained in this thesis are as follows:

(1) In order to open a new vista in the field of plasma physics, attempts were made in treating the finite-amplitude oscillations in electron plasmas. The subject is still in the active stage of development; and the results obtained are new and worth noticing in that they have contributed to the understanding of what is vaguely called the turbulent fluctuations. The author suggests that it will be a future central problem of plasma physics to introduce the ion correlations and, if possible, to take into account the influence of an external magnetic field.

(2) Diffusion of charged particles across a strong magnetic field is assessed.

(3) By introducing a particle-physics analogue into the domain of numerical analysis, a novel technique has been developed in order to find appropriate starting values for the iterative solution of non-linear algebraic equations. This is the first attempt, ever made in the field of the Monte Carlo method, for the removal of crucial difficulties in the conventional method of solving simultaneous non-linear algebraic equations.

(4) A new theory is proposed in order to account for various anomalous features of the equatorial sporadic E layers

in the ionosphere. The mechanism considered therein may also provide for the understanding of the "spread-F" phenomena, which are not very well understood as yet. The proposed theory is a rival theory against D. T. Farley's, who only considered the two-stream plasma wave instability of a uniform plasma in a magnetic field.

Acknowledgement

The author acknowledges that he is really grateful to Prof. T. Kiyono for his kind advice and constant encouragement. Thanks are also due to Prof. K. Maeda, whose constructive comments have been very helpful and illuminating.

APPENDIX 1

Let us denote

$$\xi_{\vec{k}}(0) = \xi^{0}_{\vec{k}}, \quad \xi_{\vec{k}}(t) = \xi_{\vec{k}}, \quad \text{etc.} \tag{A1}$$

Then, after an obvious algebra, we have the following equation:

$$\frac{d^2}{dt^2}\left(\xi_{\vec{k}}\xi^{*0}_{\vec{k}}\right) = -\omega_p^2\,\xi_{\vec{k}}\xi^{*0}_{\vec{k}} - \sum_{\{\vec{l}\}}\frac{(\vec{k}\cdot\vec{l})}{kl}\,|\vec{k}+\vec{l}|\,\omega_p\frac{d}{dt}\left(\eta_{-\vec{l}}\,\xi_{\vec{k}+\vec{l}}\,\xi^{*0}_{\vec{k}}\right)$$
$$- i\,\frac{d}{dt}\left(f_{\vec{k}}(t)\,\xi^{*0}_{\vec{k}}\right), \tag{A2}$$

$$\frac{d}{dt}\left(\eta_{-\vec{l}}\,\xi_{\vec{k}+\vec{l}}\,\xi^{*0}_{\vec{k}}\right) = -\omega_p\,\xi_{\vec{k}+\vec{l}}\,\xi_{-\vec{l}}\,\xi^{*0}_{\vec{k}} + \omega_p\,\eta_{-\vec{l}}\,\eta_{\vec{k}+\vec{l}}\,\xi^{*0}_{\vec{k}}$$
$$- \omega_p\sum_{\{\vec{l}'\}}\frac{\vec{k}+\vec{l}\cdot\vec{l}'}{|\vec{k}+\vec{l}|\,l'}\,|\vec{k}+\vec{l}+\vec{l}'|\,\eta_{-\vec{l}}\,\eta_{-\vec{l}'}\,\xi_{\vec{k}+\vec{l}+\vec{l}'}\,\xi^{*0}_{\vec{k}}$$
$$- i\,f_{\vec{k}+\vec{l}}(t)\,\eta_{-\vec{l}}\,\xi^{*0}_{\vec{k}}, \tag{A3}$$

$$\frac{d}{dt}\left(\xi_{-\vec{l}}\,\xi_{\vec{k}+\vec{l}}\,\xi^{*0}_{\vec{k}}\right) = \omega_p\,\xi_{-\vec{l}}\,\eta_{\vec{k}+\vec{l}}\,\xi^{*0}_{\vec{k}} + \omega_p\,\eta_{-\vec{l}}\,\xi_{\vec{k}+\vec{l}}\,\xi^{*0}_{\vec{k}}$$
$$- \sum_{\{\vec{l}'\}}\frac{\vec{k}+\vec{l}\cdot\vec{l}'}{|\vec{k}+\vec{l}|\,l'}\,|k+l+l'|\,\omega_p\,\eta_{-\vec{l}'}\,\xi_{\vec{k}+\vec{l}+\vec{l}'}\,\xi_{-\vec{l}}\,\xi^{*0}_{\vec{k}}$$
$$- \sum_{\{\vec{l}'\}}\frac{(-\vec{l})\cdot\vec{l}'}{ll'}\,|-\vec{l}+\vec{l}'|\,\omega_p\,\eta_{-\vec{l}'}\,\xi_{-\vec{l}+\vec{l}'}\,\xi_{\vec{k}+\vec{l}}\,\xi^{*0}_{\vec{k}}$$

$$- i f_{\vec{k}+\vec{l}}(t)\, \xi_{-\vec{l}}\, \xi^{*0}_{\vec{k}} - i f_{-\vec{l}}(t)\, \xi_{\vec{k}+\vec{l}}\, \xi^{*0}_{\vec{k}}, \qquad (A4)$$

$$\frac{d}{dt}\left(\xi_{-\vec{l}}\, \eta_{\vec{k}+\vec{l}}\, \xi^{*0}_{\vec{k}} \right) = -\omega_p\, \xi_{-\vec{l}}\, \xi_{\vec{k}+\vec{l}}\, \xi^{*0}_{\vec{k}} + \omega_p\, \eta_{-\vec{l}}\, \eta_{\vec{k}+\vec{l}}\, \xi^{*0}_{\vec{k}}$$

$$- \sum_{\{\vec{l}'\}} \frac{(-\vec{l})\cdot\vec{l}'}{l l'} |-\vec{l}+\vec{l}'|\, \omega_p\, \eta_{-\vec{l}'}\, \xi_{-\vec{l}+\vec{l}'}\, \eta_{\vec{k}+\vec{l}}\, \xi^{*0}_{\vec{k}}$$

$$- i f_{-\vec{l}}(t)\, \eta_{\vec{k}+\vec{l}}\, \xi^{*0}_{\vec{k}}, \qquad (A5)$$

$$\frac{d}{dt}\left(\eta_{-\vec{l}}\, \eta_{\vec{k}+\vec{l}}\, \xi^{*0}_{\vec{k}} \right) = -\omega_p\, \xi_{-\vec{l}}\, \eta_{\vec{k}+\vec{l}}\, \xi^{*0}_{\vec{k}}$$

$$- \omega_p\, \eta_{-\vec{l}}\, \xi_{\vec{k}+\vec{l}}\, \xi^{*0}_{\vec{k}}, \qquad (A6)$$

$$\frac{d}{dt}\left(\eta_{\vec{k}}\, \xi^{*0}_{\vec{k}} \right) = -\omega_p\, \xi_{\vec{k}}\, \xi^{*0}_{\vec{k}}. \qquad (A7)$$

We neglect the terms containing f and take averages of the above equations in the sense expounded in the earlier paragraph of §1.1, making use of the property (1.57) at the same time. Hence we have Eqs.(1.58), (1.59) and (1.60).

APPENDIX 2

We attach superscripts o to the quantities at time t=0, and none to those at time t=t. In the following, all the suffix vectors of ρ and π are non-zero, namely, ρ and π are all fluctuating quantities. By the same method as in the preceding chapter, the equations of motion from Eq.(2.4)

$$d\rho_{\vec{k}}/dt = -\frac{1}{m}\sum_{\substack{\vec{h} \\ \vec{k}+\vec{h}\neq 0}} (\vec{k}\cdot\vec{h})\,\pi_{\vec{h}}\,\rho_{\vec{k}+\vec{h}} + \frac{N}{m}k^2\,\pi_{-\vec{k}}\,, \qquad (A8)$$

$$d\pi_{\vec{k}}/dt = \frac{1}{2m}\sum_{\vec{h}} (\vec{k}-\vec{h})\cdot\vec{h}\;\pi_{\vec{k}-\vec{h}}\,\pi_{\vec{h}} - \frac{4\pi e^2}{k^2}\rho_{-\vec{k}} \qquad (A9)$$

yield the following system of equations ($\vec{k}$ is simplified to k, as long as there is no confusion of symbols):

$$\frac{d}{dt}\langle \rho_k\, \rho_k^{*o}\rangle = -\frac{1}{m}\sum_h (k\cdot h)\langle \pi_h\, \rho_{k+h}\, \rho_k^{*o}\rangle + \frac{1}{m}k^2 N\langle \pi_{-k}\, \rho_k^{*o}\rangle,$$

$$\frac{d}{dt}\langle \pi_h\, \rho_{k+h}\, \rho_k^{*o}\rangle = \frac{1}{2m}\sum_l (h-l)\cdot l\,\langle \pi_{h-l}\,\pi_l\, \rho_{k+h}\, \rho_k^{*o}\rangle - \frac{4\pi e^2}{h^2}\langle \rho_{-h}\, \rho_{k+h}\, \rho_k$$

$$\frac{d}{dt}\langle \rho_{-h}\, \rho_{k+h}\, \rho_k^{*o}\rangle = \frac{1}{m}\sum (h\cdot l)\langle \pi_l\, \rho_{-h+l}\, \rho_{k+h}\, \rho_k^{*o}\rangle + \frac{N}{m}k^2\langle \pi_h\, \rho_{k+h}\, \rho_k^{*}$$

$$-\frac{1}{m}\sum_l (k+h)\cdot l\,\langle \pi_l\, \rho_{k+h+l}\,\rho_{-h}\, \rho_k^{*o}\rangle + \frac{N}{m}(k+h)^2\langle \pi_{-k-h}\,\rho_{-h}\, \rho_k^{*o}$$

$$\frac{d}{dt}\langle \pi_{-k-h}\pi_h \rho_k^{*o}\rangle = -\frac{1}{2m}\sum (k+h+\ell)\cdot\ell \langle \pi_{-k-h-\ell}\pi_\ell\pi_h\rho_k^{*o}\rangle$$

$$-\frac{4\pi e^2}{(k+h)^2}\langle \pi_h \rho_{k+h}\rho_k^{*o}\rangle + \frac{1}{2m}\sum_\ell (h-\ell)\cdot\ell \langle \pi_{h-\ell}\pi_\ell\pi_{-k-h}\rho_k^{*o}\rangle$$

$$-\frac{4\pi e^2}{h^2}\langle \pi_{-k-h}\rho_{-h}\rho_k^{*o}\rangle ,$$

$$\frac{d}{dt}\langle \pi_{-k}\rho_k^{*o}\rangle = \frac{1}{2m}\sum_r (-k-r)\cdot r \langle \pi_{-k-r}\pi_r\rho_k^{*o}\rangle - \frac{4\pi e^2}{k^2}\langle \rho_k \rho_k^{*o}\rangle .$$

An assumption is made that $\langle \pi(t)\rho(t)\rangle \sim 0$ (but, $\langle \pi(t)\rho(0)\rangle$ need not be zero), $\langle \rho_{\vec{k}'}(t)\rho^*_{\vec{k}''}(t)\rangle = 0$ ($\vec{k}' \neq \vec{k}''$) and $\langle \pi_{\vec{k}'}(t)\pi_{\vec{k}''}(t)\rangle = 0$ ($\vec{k}' \neq \vec{k}''$). If we decompose the fourth order correlations as in Eq.(1.56) of the foregoing chapter, then the above equations are simplified as shown in Eq.(2.44).

APPENDIX 3

Here we perform a Laplace transformation on Eq.(2.44). [Eq.(2.44) is a linear equation and the simultaneous correlations are treated as constants. To find these constants is one of the points of the present chapter.] Notice Eq.(2.39).

After laborious algebra, we have the following equations

$$\Big[s\Delta(s) + \frac{1}{m}\sum_{h}(k\cdot h)D_{h,k}(s) + \Big\{\frac{1}{m}\sum_{h}(k\cdot h)E_{h,k}(s) - \frac{N}{m}k^2\Delta(s)\Big\}$$

$$\times B'_k(s)\Big] L[\langle \rho_k \rho_k^{*0}\rangle]$$

$$= \Big\{-\frac{1}{m}\sum_{h}(k\cdot h)E_{h,k}(s) + \frac{N}{m}k^2\Delta(s)\Big\} C'_k(s)$$

$$-\frac{1}{m}\sum_{h}(k\cdot h)\frac{N}{m}(k+h)^2 s^2\langle \pi_{-k-h}\pi_h \rho_k^{*0}\rangle + \langle \rho_k \rho_k^{*0}\rangle\Delta(s), \tag{A10}$$

where L is the operator for the Laplace transformation.

$$\Delta(s) = s^2(s^2 + 4\omega_p^2), \tag{A11}$$

$$B'_k(s) = \Big[\frac{4\pi e^2}{2m^2\Delta(s)}\sum_{r}(k+r)^2 r^2 s^2\Big\{\frac{\langle \pi_{-r}\pi_r\rangle}{(k+r)^2} + \frac{\langle \pi_{-k-r}\pi_{k+r}\rangle}{r^2}\Big\} - \frac{4\pi e^2}{k^2}\Big]$$

$$\times\Big[s - \frac{1}{2m}\sum_{r}(k+r)\cdot r\frac{1}{\Delta(s)}\Big[2s\frac{4\pi e^2}{(k+r)^2}\cdot\frac{4\pi e^2}{r^2}\cdot\frac{1}{m}\Big\{(r\cdot k)\langle \rho_{k+r}\rho^*_{k+r}$$

$$-(k+r)\cdot k\langle \rho_r \rho_r^*\rangle\Big\} + s(s^2+2\omega_p^2)\frac{1}{2m}\Big(\sum_{l} l^2\langle \pi_{-l}\pi_l\rangle$$

$$- 2(r\cdot k)\langle \pi_{-r}\pi_r\rangle + (k+r)\cdot r\,\langle \pi_{-(k+r)}\pi_{k+r}\rangle)\Big]\Big]^{-1},$$

$$C'_k(s) = \Big[-\frac{1}{2m}\cdot\frac{1}{\Delta(s)}\sum_r (k+r)\cdot r s\,(s^2+2\omega_p^2)\langle \pi_{-k-r}\pi_r \rho_k^*\rangle\Big]$$

$$\times\Big[\,s - \frac{1}{2m}\sum_r (k+r)\cdot r\,\frac{1}{\Delta(s)}\,[\cdots]\,\Big]^{-1}$$

(The contents of $[\,s - (1/2m)\sum (k+r)\cdot r\,(1/\Delta(s))\,[\cdots]\,]^{-1}$ above are the same as that of $B'_k(s)$.),

$$D_{h,k}(s) = \frac{(k+h)\cdot h}{m}\Big\{s(s^2+2\omega_p^2)\langle \pi_{-h}\pi_h\rangle - 2\omega_p^2 s\,\frac{(h+k)^2}{h^2}$$

$$\times\langle \pi_{-h-k}\pi_{h+k}\rangle\Big\},$$

$$E_{h,k}(s) = \frac{4\pi e^2}{h^2}s^2\frac{1}{m}\Big\{(h\cdot k)\langle \rho_{k+h}\rho^*_{k+h}\rangle - (h+k)\cdot k\langle \rho_h\rho_h^*\rangle\Big\}$$

$$- \frac{N}{m}(k+h)^2 s^2\frac{1}{2m}\Big\{\sum_l l^2\langle \pi_{-l}\pi_l\rangle - 2(h\cdot k)\langle \pi_{-h}\pi_k\rangle$$

$$+ 2(h+k)\cdot k\langle \pi_{h+k}\pi_{-h-k}\rangle\Big\}.$$

Next we solve Eq.(A10) to obtain an expression for $L[\langle \rho_k \rho_k^{*0}\rangle]$, and equate the denominator of the rational equation to zero. Thus we obtain an algebraic equation whose roots are the matter of concern. We have supposed that the oscillator system is a system of weakly-coupled oscillators (as justified experimentally), so that we may neglect the square of $\sum$ [square of amplitudes] in comparison with $\sum$ [square of amplitudes]. We have several roots of the algebraic equation, out of which we choose the one which leads to the branch of ω_p in the limit of linear approximation (namely, the amplitudes $\rightarrow$ 0). In order to find the shift of frequency corresponding to Eq.(2.46), we select out only the specific terms from the sum $\sum$, as stated immediately below Eq.(2.46). The branch of $\omega(k)$ tending to $\pm\omega_p$ in the limit of zero amplitude is given by a type of equation:

$$2s^2 = -C_1 + \sqrt{C_1^2 - 4C_2}\,. \qquad (A12)$$

For a one-dimensional model, the actual expression for Eq.(A12) is as follows:

$$\begin{aligned}
2s^2 &= -2\,\omega^2(k) \\
&= -2\omega_p^2 + \frac{2}{3}\cdot\frac{1}{m}\,k\,(k+\ell)\,\mathcal{E}(\ell) + \frac{2}{3}\cdot\frac{1}{4m}\,\ell^2\mathcal{E}(\ell) \\
&\quad - \frac{1}{3}\cdot\frac{1}{4m}\,k^2\mathcal{E}(\ell) + \frac{2}{3}\cdot\frac{1}{2m}\,k\cdot\ell\;\mathcal{E}(k+\ell) \\
&\quad + \left(-\frac{4}{3}\,k\cdot(k+\ell) + \frac{1}{3}\,k^2\right)\frac{1}{m}\left\{\mathcal{E}(\ell) - \mathcal{E}(k+\ell)\right\} \\
&\quad - \frac{1}{3m}\left\{\left(\frac{\ell(k+\ell)}{k} - 2(k+\ell)^2\right)\mathcal{E}(\ell) + 2\ell(k+\ell)\,\mathcal{E}(k+\ell)\right\}.
\end{aligned} \qquad (A13)$$

We know approximately

$$\omega^2(k) - \omega_p^2 \cong 2\omega_p \Delta_2 \omega(k), \qquad (A14)$$

so that when $|l|$ is large enough compared with $|k|$,

$$\mathcal{E}(k+l) \sim \mathcal{E}(l);$$

from the above relations we finally have

$$\Delta_2 \omega(k) \cong \frac{1}{12\omega_p m} \cdot \frac{l^3}{k} \mathcal{E}(l). \quad (|k| \ll |l|) \qquad (A15)$$

APPENDIX 4

(LETTER TO THE EDITOR)

Addenda to "A Theory of the Turbulent Electric Field Fluctuations in Electron Plasmas II"
[T. Tsuda, Prog. Theor. Phys. 30 (1963), 17]

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Concerning the paper quoted above, it has come to the author's notice that he should spend a few more words in order to avoid misunderstanding about a point of particular importance in the paper.

In the equation immediately below Eq. (47) of the quoted paper, we have shown that

$$\mathcal{E}(k)\,\frac{\Delta_2\omega(k)}{\omega_p} = \mathcal{E}(l)\,\frac{\Delta_2\omega(l)}{\omega_p}\,, \qquad (a)$$

where $\Delta_2\omega(k)$ and $\Delta_2\omega(l)$ are the frequency-shifts, due to an interaction Hamiltonian

$$\delta H^I_{k,l} = -\frac{1}{2m}(k\cdot l)\pi_k\pi_l\rho_{k+l} - \frac{1}{2m}(k-l)\cdot l\,\pi_{k-l}\pi_l\rho_k\,, \qquad (b)$$

of the waves with wave numbers k and l respectively; $\mathcal{E}$(k) and $\mathcal{E}$(l) are respectively the energies attributable to the modes with wave numbers k and l. It is insinuated, but not demonstrated there in the paper that the right-hand side of Eq. (a), which is denoted by K(l), is independent of the smaller wave number k.

This, however, constituted a very essential part of the discussions determining the spectral distribution which followed; therefore the author would like to justify the intuitive argument and avoid misunderstanding on the side of the readers.

In deriving Eq. (A6) in the Appendix II of the quoted paper, no assumption was introduced concerning the asymmetry in the interaction between waves with wave numbers k and l ($|\mathbf{k}| \ll |\mathbf{l}| < \mathbf{k}_0$). Therefore, interchanging k with l, we have an explicit expression for $\Delta_2\omega(l)$, i. e.

$$\Delta_2 \omega(l) = -\frac{1}{4m\omega_p}\left\{ l^2 \left(1 + O(k/l)\right) \mathcal{E}(l) + \cdots \right\}. \tag{c}$$

It is thus demonstrated that, insofar as $|l| \gg |k|$, the right-hand side of Eq. (a) in the above is independent of k to the first order of approximation. This reflects the fact that, in turbulent fluctuations, small-scale components are not much controlled by large-scale fluctuations, while the converse does not hold true.

In this way we have

$$K(l) \equiv \mathcal{E}(l)\,\frac{\Delta_2\omega(l)}{\omega_p} \cong -\frac{l^2\,\mathcal{E}^2(l)}{4m\omega_p^2}\,. \tag{d}$$

Furthermore let it be notified that we cannot distinguish between waves with wave number vectors +k and -k (see Eq. (5)); and waves with $\pm$ k interact with waves with $\pm$ l through the interaction Hamiltonian $\delta H^{I}_{\pm k, \pm l}$. The frequency-shifts arising from this interaction are such that they conform to Eqs. (a) and (d); hence the shifts are always negative.

If we add up the contributions due to various l ($|l| \gg |k|$), then

$$\sum_{\substack{l \\ |l| \gg |k|}} \Delta_2 \omega(k) \cong -\frac{1}{12\omega_p m} \frac{1}{|k|} \sum_{\substack{l \\ |l| \gg |k|}} |l|^3 \mathcal{E}(l) ,$$

$$\mathcal{E}(k) \sum \Delta_2 \omega(k) / \omega_p = \sum \mathcal{E}(l) \Delta_2 \omega(l) = \sum K(l).$$

$$\therefore \quad \mathcal{E}(k) = 12\, m\, \omega_p^2 \frac{\sum |K(l)|}{\sum |l|^3 \mathcal{E}(l)} \cdot |k| . \qquad (e)$$

When $\mathcal{E}(l) \propto |l|$ ($k_o \ll |l| < k_c$), then $K(l)/[l^3 \mathcal{E}(l)]$ is a constant irrespective of l; hence we have

$$\frac{\sum |K(l)|}{\sum |l|^3 \mathcal{E}(l)} = \frac{|K(l)|}{|l|^3 \mathcal{E}(l)} ,$$

which, together with Eq. (e), yields Eq. (48) of the paper in question.